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The medieval palace that predates a modern geometrical discovery.

The Alhambra palace in the Spanish city of Granada (top picture opposite) was constructed in the 13th century by the Nasrid king Mohammed ibn Yusuf ben Nasr. That fact by itself is not surprising as there are several such medieval palaces in Southern Spain, notably in Sevilla and Cordoba. What makes the Alhambra stand out from a mathematical perspective is the presence of all the 17 wallpaper groups (https://bit.ly/2MVTE28) in the motifs and designs on it walls. In simple terms a wallpaper group is an infinitely repeatable pattern in two dimensions; these are often called tessellations. The first proof that there are just 17 different types of wallpaper group or tessellation started appearing from the late 19th century onwards. Tessellations have either no rotational symmetry or rotational symmetry of orders 2, 3, 4 or 6. The interesting tessellations have rotational symmetry of orders 2, 3, 4 or 6 and segments of ones in the Alhambra corresponding to these are shown in the 4 images opposite (clockwise from the top left). The modern day artist who popularised the tessellations found in the Alhambra was M. C. Escher. An interesting mathematical field trip for sixth form mathematics students would be to visit the Alhambra and identify the 17 types of tessellation.

Perhaps a convincing explanation of why $-1 \times -1 = 1$?

The problem of explaining why $-1 \times -1 = 1$ is one that vexes both teachers and students. Of course there is a formal proof of this using field axioms but this is of limited use in the classroom where an 'organic' explanation is more relevant. For example one can give an 'organic' explanation of why $3 \times 4 = 12$ by arranging a set of 4 rows each containing 3 objects and counting the whole set. Informally speaking what we seek is an 'organic' explanation of why $- \times - = +$. The 'organic' explanation that I have encountered that could possibly meet with approval in the classroom is the following:

1. The positive direction is walking forwards. So let walking forwards be signified by $'+'$.
2. The negative direction is walking backwards. So let walking backwards be signified by $'-'$.

This seems practically sensible and few would argue against this definition. We can also equivalently say that:

3. $'+'$ is a video recording played forwards.
4. And $'-'$ is a video recording played in reverse.

In this context when any two of the four actions above are followed one after the other we say that the two actions are multiplied $'\times'$.

So for example if we forwards play a video recording of a person walking forwards we are exhibiting $+ \times +$. And since the video shows a person walking forwards we can conclude that $+ \times + = +$.
What happens if a video recording of a person walking backwards is played in reverse? This is exhibiting $- \times -$. Since the video will show a person walking forwards we may conclude that $- \times - = +$.

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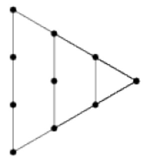
Formulae for triangle numbers and their equivalents in 3 and higher dimensions.

The well known sequence of *triangular* numbers is given by the diagram:

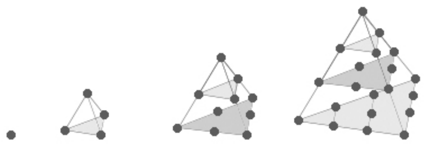


The sequence is derived by counting successively the points in the first (trivial) triangle, the second triangle, and so on. The sequence is: 1, 3, 6, 10,........

The sequence can be re-written as the successive sums of the natural numbers:

1, (1+2), (1+2+3), (1+2+3+4),.....

In three dimensions the analogue is the sequence of *tetrahedral* numbers given by the diagram:



As one can see the sequence is formed by adding successive *triangular* numbers:

1, 1+3, 1+3+6, 1+3+6+10, ...

Formally, being located in 2 dimensions, the triangular numbers are denoted by $P_2(m)$, where m denotes the sequence placement. So

$P_2(1) = 1, P_2(2) = 3, P_2(3) = 6, \ldots\ldots..$

The tetrahedral numbers are denoted by $P_3(m)$,

$P_3(1) = 1, P_3(2) = 4, P_2(3) = 10, P_2(4) = 20, \ldots\ldots..$

The letter P stands for *polytope*, the generalisation of triangles in higher dimensions.

The formula $P_2(m) = \frac{1}{2}m(m+1)$ is well known in school mathematics. Evidently $P_3(m) = P_2(1) + P_2(2) + \ldots.. + P_2(m)$. That is, the m^{th} tetrahedral numbers is the sum of the first m triangular numbers. Deriving a formula for $P_3(m)$ is considered geometrically in the University of Leicester/AMSP y12 advanced problem solving classes, but it can also be found algebraically if one knows the formula for the sum of the first m squares; namely the formula

$\sum_{r=1}^{m} r^2 = \frac{1}{6}m(m+1)(2m+1)$. Then

$$\begin{aligned} P_3(m) &= \sum_{r=1}^{m} P_2(r) \\ &= \sum_{r=1}^{m} \frac{1}{2}r(r+1) \\ &= \frac{1}{2}\left(\sum_{r=1}^{m} r^2 + \sum_{r=1}^{m} r\right) \end{aligned}$$

Putting $\sum_{r=1}^{m} r^2 = \frac{1}{6}m(m+1)(2m+1)$ and $\sum_{r=1}^{m} r = \frac{1}{2}m(m+1)$ in the above and simplifying will give us:

$P_3(m) = \frac{1}{6}m(m+1)(m+2)$.

Finding a formula for $P_4(m)$, the analogue of the triangular numbers in 4 dimensions, requires finding the sum $\sum_{r=1}^{m} P_3(r)$, which algebraically requires a formula for $\sum_{r=1}^{m} r^3$. Subsequent *polytope* numbers will increase the degree of algebraic difficulty.

Nevertheless is one examines the formulae $P_2(m) = \frac{1}{2}m(m+1)$ and $P_3(m) = \frac{1}{6}m(m+1)(m+2)$ one can conjecture that

$P_4(m) = \frac{1}{4!}m(m+1)(m+2)(m+3)$ and
$P_n(m) = \frac{1}{n!}m(m+1)\ldots(m+n-1)$.

The derivation of the formula for $P_n(m)$ is part of problem 64 in the **y13 challenge problems** document found in the outreach webpage. **Only** if you have done the y13 binomial expansion and relish a challenge, the steps to derive the formula for $P_n(m)$ that bypasses the need to find $\sum_{r=1}^{m} r^k, k \geq 3$, is given below. Any reader who submits a full solution to problem 64 will be acknowledged in the newsletter number 8.

1. Consider the function $\mathrm{f}(m,n)$ defined on pairs of non-negative integers which satisfies the following:

$\mathrm{f}(m,n) = \mathrm{f}(m-1,n) + \mathrm{f}(m,n-1)$, when both m and n are positive.
$\mathrm{f}(m,0) = 1$, when $m \geq 0$ and $\mathrm{f}(0,n) = 0$, when $n \geq 1$.

One can find that
$\mathrm{f}(m,1) = m$,
$\mathrm{f}(m,2) = \frac{1}{2}m(m+1) = P_2(m)$,
$\mathrm{f}(m,3) = \frac{1}{6}m(m+1)(m+2) = P_3(m)$,
and that subsequently, for $n \geq 3$, $\mathrm{f}(m,n) = P_n(m)$.

2. Show that $\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \mathrm{f}(m,n)x^m y^n = (1-y)(1-x-y)^{-1}$, whenever $-1 < x+y < 1$. To prove this set $S = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \mathrm{f}(m,n)x^m y^n$ and then show that $S = 1 + xS + yS - y$ (This will need some dogged determination!).

3. Find the coefficient C of $x^m y^n$ in the binomial expansion of
$(1-y)(1-(x+y))^{-1}$.

With some care you should find that
$C = {}^{m+n}C_n - {}^{m+n-1}C_{n-1}$
$= \dfrac{(m+n)!}{m!n!} - \dfrac{(m+n-1)!}{m!(n-1)!}$
$= \ldots. = \dfrac{m(m+1)\ldots(m+n-1)}{n!}$.

Since the coefficient of $x^m y^n$ on either side of

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \mathrm{f}(m,n)x^m y^n = \frac{1-y}{1-x-y}$$

is the same you can conclude that

$$\mathrm{f}(m,n) = \frac{m(m+1)\ldots(m+n-1)}{n!}$$

Or $P_n(m) = \dfrac{m(m+1)\ldots(m+n-1)}{n!}$.

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