The mathematics in a cauliflower.

The picture opposite is that of a Romanesco cauliflower grown by the author. Besides being a tasty vegetable its appearance reveals some mathematics.
Specifically the Romanesco cauliflower approximately exhibits the property of endless repeating patterns in its structure. In each floret of this cauliflower there are identical sub-florets and within each sub-floret there are identical sub-sub-florets and so on. In other words the Romanesco cauliflower is an approximation of a *fractal*. This cauliflower is an approximation of a fractal because the pattern cannot endlessly repeat to a physically microscopic level. For more on the fractal property of the Romanseco cauliflower see: `https://www.fourmilab.ch/images/Romanesco/`.

All the ones.

Consider the first four terms of a sequence of numbers:

$3^2 + 2 = 11$
$33^2 + 22 = 1111$
$333^2 + 222 = 111111$
$3333^2 + 2222 = 11111111$

Does this pattern continue or is this just a feature of the first few terms of the sequence?

That is, does $333...3^2 + 222...2$, where there are n 3's and n 2's, equal $111......11$, where there are $2n$ 1's?

Clearly we cannot confirm or deny the conjecture by machine calculation so we must resort to proving it in some other way.

Rather than compute the digits of $333...3^2 + 222...2$ we consider the easier computation of 9 times this number. Namely $N = 999...9^2 + 9 \times 222...2$, where there are n 9's and n 2's

Now it is not difficult to see (i.e. to be convinced by the fact) that

$9 \times 222...2 = 1999..98$, where there are $(n-1)$ 9's.

Now $999...9^2 = (1000...0 - 1)^2$, where there are n 0's. And:

$(1000...0 - 1)^2$; n 0's.
$= 100000......00 - 2000....0 + 1$; $2n$ 0's in the 1st term and n 0's in the 2nd.
$= 999..98000...0 + 1$; $(n-1)$ 9's and n 0's.
$= 999..98000..01$; $(n-1)$ 9's and $(n-1)$ 0's.
Note that there are $2n$ digits here.

To this number we need to add $9 \times 222...2 = 1999...98$, where there are $(n-1)$ 9's.

So now we can determine the digits in $N = 999..98000..01 + 1999..98$. Remember $999..98000..01$ has $(n-1)$ 9's and $(n-1)$ 0's and $1999..98$ has $(n-1)$ 9's.

$$N = 999..98000...01+ \\ 1999...98$$

Here the 0's in the first number and the 9's in the second line up. As do the digits 8 and 1. Thus

That is, $N = 999999......99$, where there are $2n$ digits.

Since $N = 9 \times (333...3^2 + 222...2)$, where there are n 3's and n 2's, we see that $333...3^2 + 222...2$ equals:

$111111......11$, where there are $2n$ 1's. So the conjecture is **true**.

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Small sums can sometimes be big.

Clearly the sum of the first n natural numbers is less than the sum of the next n natural numbers. That is,

$1 + 2 + 3 + ... + n$
$< (n + 1) + (n + 2) + ... + 2n.$

What are the two sums? To determine each, as in the last newsletter, we need to remind ourselves that the sum of an arithmetic sequence $a, a + d, a + 2d, ..., a + (n - 1)d$ is $S_n = \frac{1}{2}n(2a + (n - 1)d)$.

The left sum is therefore

$L = \frac{1}{2}n(n + 1)$

And the right

$R = \frac{1}{2}n(2(n+1)+n-1) = \frac{1}{2}n(3n+1).$

It can easily be verified that $R - L = n^2$, so that the right sum is n^2 bigger than the left one.

This fact can also (much more easily) be verified by comparing the terms of each sum:

1	2	.	.	n
$n+1$	$n+2$	.	.	$2n$

Each of the n terms in the second row is n bigger than the corresponding ones in the first, so this implies that the sum of the terms in the second row is n^2 bigger than the sum of terms in the first.

This would also apply if we started summing from any number $m + 1$ instead of 1, as implied by the table below:

$m+1$	$m+2$	.	$m+n$
$m+n+1$	$m+n+2$	.	$m+2n$

To give the left sum L any chance of equalling or surpassing the right R we could make L consist of $n + 1$ consecutive numbers and R of the n subsequent consecutive ones. So let us see if:

$L = 1 + 2 + 3 + ... + n + (n + 1)$
$> R = (n+2)+(n+3)+....+2n+1?$

Using the arithmetic sum formula we have

$L = \frac{1}{2}(n + 1)(2 + n).$
$R = \frac{1}{2}n(2(n + 2) + (n - 1)).$

Simplifying $R = \frac{3}{2}n(n + 1)$.

And then, after some algebra,

$L - R = -n^2 + 1.$

So L never exceeds R.

So now let us investigate the cases where the consecutive numbers do not start at 1. Will R always be bigger than L? Is there any case where the two will be equal? Are there any cases where L is bigger than R?

Specifically we examine the sum of $n + 1$ consecutive numbers starting with the number $M > 1$ and the sum subsequent n consecutive numbers starting with $M + n + 1$.

Explicitly the numbers involved are:

$M, M + 1, M + 2, ..., M + n$, for the left sum.

$M + n + 1, M + n + 2, ..., M + n + n$, for the right.

We have for the left sum:

$L = M(n + 1) + (1 + 2 + 3 + ... + n)$

Or $L = Mn + M + \frac{1}{2}n(n + 1)$

For the right sum:

$R = (M + n)n + (1 + 2 + 3 + ... + n)$

Or $R = Mn + n^2 + \frac{1}{2}n(n + 1)$

The difference of the two sums is therefore:

$R - L = (Mn + n^2) - (Mn + M) = n^2 - M$. This implies 3 cases:

1. $R > L$ whenever $n^2 > M$. An example of this case was discussed earlier with $M = 1$. In this case R is always $n^2 - M$ bigger than L.
2. $R = L$ whenever $n^2 = M$. Here *interestingly the numbers are those between two successive squares*, excluding the bigger square. This is implied by the sums:

   $L = n^2+(n^2+1)+.....+(n^2+n)$

   $R = (n^2+n+1)+.....+(n^2+2n)$

   Note $(n^2 + 2n) = (n + 1)^2 - 1$.

   A few examples of this are:

   $$1 + 2 = 3$$

   $$4 + 5 + 6 = 7 + 8$$

   $$9 + 10 + 11 + 12 = 13 + 14 + 15$$

   $$16+17+18+19+20 = 21+22+23+24$$

   $$....$$

   $625 + 626 + ... + 650 + 651$
   $= 652 + 653 + ... + 674 + 675$
3. $L > R$ whenever $n^2 < M$. In this case the left sum is $M - n^2$ bigger than the right.

   So, for example,

   $17 + 18 + 19 + 20 + 21 = 95 > 22 + 23 + 24 + 25 = 94$

   Here $M = 17$ and $n = 4$, so that $M - n^2 = 1$.

   $650+651+......+675 = 17225 > 676 + 677 + .. + 700 = 17200$

   Here $M = 650$ and $n = 25$, so that $M - n^2 = 25$.

The reader might like to investigate the cases where L consists of $n + k$ consecutive numbers, $k = 2, 3, ...$, and R of the next n consecutive numbers.

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