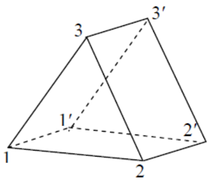
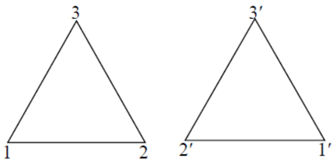
The Department of Mathematics, Schools Newsletter, No.6





The mathematics of a child's shape sorter.

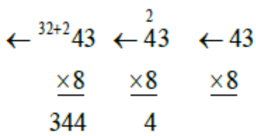
It is often said, and with some justification, that children's toys nowadays are educational. In the case of a shape sorter this is certainly true as a child playing with one develops manual dexterity and shape recognition amongst other skills. However a primary school student playing with a child's shape sorter can also precisely enumerate the number of rotational symmetries of the solid shapes available with the shape sorter. This is done by simply counting the number of different ways a particular shape can posted through the relevant entry point.

For example, if the equilateral triangular prism shown is posted through the equilateral triangle hole in the child's shape sorter then there are 6 different ways of posting. More precisely with the end 123 facing us we have 3 ways of posting and these 3 ways represent the three 120^0 rotational symmetries about the axis that passes through the centres of the triangular faces. And with the end 1'2'3' facing us we have another 3 ways of posting and these correspond to the three 90^0 rotational symmetries about the 3 axes which pass through the centre of each rectangle and the centre of of the opposite edge. This gives us the 6 rotational symmetries. Not only can this procedure be extended to count the rotational symmetries of a broad range of shapes, but a further analysis shows that the posting procedure is an example of the deep *Orbit-Stabilizer* theorem (for justifications of all the above claims see http://bit.ly/13MXmxW).

Sinister aspects of arithmetic. Firstly by sinister we just mean left-handed as in the original sense of the Latin word. English writing is sinister as it proceeds from the *left to right*. This *left to right* linguistic (or even, cultural) norm is partially extended to numbers and its arithmetic. For example in a ruler or number line smaller numbers increase *left to right*: 1, 2, 3, 4, ... But this is a partial norm because when we represent large numbers using place order. For example, 4736 does not mean 4 and 70 and 300 and 6000. It is the other way around *right to left*:

10^3	10^2	10^1	10^0
4	7	3	6

Next consider how multiplication is processed. We will multiply 43 and 8:



The processing is clearly *right to left*. All this is, of course, accords with the *right to left* writing orientation in Arabic. This is perhaps not a surprise as it is well known now that medieval Arab scholars, notably Al Kindi and Al Khwarizmi, transmitted the Hindu numerals and associated decimal arithmetic to the Middle East. From there the numerals and decimal arithmetic were transmitted westwards. It is for this reason the number system and arithmetic that is universally used nowadays is called Indo-Arabic numbers and arithmetic. A good history of this phenomena is to be found in this history of mathematics webpage of St Andrews university: http://bit.ly/1RYQj1x.

Note that long division is undertaken in the *left to right* orientation. However this is because division is the inverse process of multiplication. Remember when we put on our footwear we first put on socks and then shoes, but when we invert the process; i.e. to take them off we must first remove shoes and then socks: everything is reversed.

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A useful medieval algebraic trick.

In medieval times mathematicians explored series expansions of certain quantities, for example π. In this article we describe one medieval exploratory method (communicated to me in 1998 by Dr J K John). And this may be firstly illustrated by manipulations of the expression

$$t = \frac{a}{1-p} \quad ....(1)$$

where a and p are such that $0 < p < 1$ and $a > 0$. Now (1) re-arranged is $t = a + pt$. In the **right** side of this we (repeatedly) replace ***t*** by $t = a + pt$ as follows

$$\begin{aligned} t &= a + p\boldsymbol{t} \\ &= a + p(a + p\boldsymbol{t}) \\ &= a + ap + p^2\boldsymbol{t} \\ &= a + ap + p^2(a + p\boldsymbol{t}) \\ &= a + ap + ap^2 + p^3\boldsymbol{t} \\ &= . \\ &= . \\ &= a + ap + ap^2 + ... + ap^{n-1} + p^n\boldsymbol{t}. \end{aligned}$$

We now have

$$t = a + ap + ap^2 + ... + ap^{n-1} + p^n t.$$

This can be re-arranged as $t\,(1 - p^n) = a + ap + ap^2 + ... + ap^{n-1}$.

Then using the fact (1) that $t = \dfrac{a}{1-p}$ we get:

$$\frac{a\,(1-p^n)}{1-p} = a + ap + ap^2 + ... + ap^{n-1}.$$

This is the well known *geometric series sum formula*.

In the event we continue the process indefinitely we see that the residual term $p^n t$ will converge to 0, so that

$$t = a + ap + ap^2 + ... + ap^n + ...$$

Or $\dfrac{a}{1-p} = a + ap + ... + ap^n + ...$

The infinite *geometric series formula*.

This exhibits a simple case of finding an infinite series decomposition of a particular sum using this medieval technique.

Next we consider how the technique can be used to identify certain algebraic factorisations.

Let $t = \dfrac{1}{x-y}$, where $x \neq y$.

We re-write this as

$$\boldsymbol{tx} = 1 + ty \quad .....(2)$$

Now multiply (2) by x and in the resultant **right** side replace ***tx*** by $tx = 1 + ty$. That is,

$$\begin{aligned} tx^2 &= x + y\boldsymbol{tx} \\ &= x + y(1 + ty) \\ &= x + y + ty^2 \end{aligned}$$

Rearranging gives $t(x^2 - y^2) = x + y$. Remembering what t is gives the familiar factorisation

$$x^2 - y^2 = (x + y)(x - y)$$

Now repeat the process: multiply $tx^2 = x + y + ty^2$ by x and in the resultant **right** side replace ***tx*** by $tx = 1 + ty$. We will get:

$$tx^3 = x^2 + xy + y^2 + ty^3$$

or $t(x^3 - y^2) = x^2 + xy + y^2$.

Remembering what t is gives the familiar factorisation

$$x^3 - y^3 = (x - y)(x^2 + xy + y^2).$$

Repeating the entire process again will yield the difference of two fourth powers:

$$x^4 - y^4 = (x - y)(x^3 + x^2y + xy^2 + y^3).$$

In general, for any integer $n > 0$,

$$x^n - y^n$$
$$= (x - y)(x^{n-1} + x^{n-2}y + ... + y^{n-1}).$$

If we begin with $t = \dfrac{1}{x+y}$ then the procedure above does not always give the expected sum of n^{th} powers factorisatons. Let us see why:

$$\begin{aligned} tx^2 &= x - y\boldsymbol{tx} \\ &= x - y(1 - ty) \\ &= x - y + ty^2 \end{aligned}$$

Re-arranging this and substituting for t just gives the difference of two squares formula derived before:

$$x^2 - y^2 = (x - y)(x + y)$$

However repeating the process gives:

$$tx^3 = x^2 - xy + y^2 - ty^3$$

or $t(x^3 + y^2) = x^2 - xy + y^2$.

Substituting for t gives the sum of two cubes formula:

$$x^3 + y^3 = (x + y)(x^2 - xy + y^2).$$

The next iteration of the process will just result in the already derived difference of fourth powers formula. However if we continue in this way we will be able to find the sum of $(2n + 1)^{\text{th}}$ powers factorisation every other iteration:

$$x^{2n+1} + y^{2n+1}$$
$$= (x + y)(x^{2n} - x^{2n-1}y + ... + y^{2n-1}).$$

While all of these derived formuale appear independent, the reality is that all of them are just special cases of the sum of a geometric series formula. Can you see how?

An impossible problem?

In 1545 the Italian mathematician Gerolamo Cardano published his seminal text on algebra entitled *Ars Magna*. In this book he posed the problem "Divide 10 into two parts so that its product is 40". If we call one part a, then the other is $10 - a$. So Cardono's problem is to find the number a such that $(10 - a)a = 40$.

An inspection of the resultant quadratic equation shows that no such real number a exists. In pure mathematics though there is a theorem that states that every polynomial equation of degree n has n solutions. As the equation $(10 - a)a = 40$ is of degree 2, there are two solutions (which obviously are not real) but which are $a = 5 \pm \sqrt{-15}$. In this sense Cardano was amongst the first to recognise the existence of complex numbers.

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