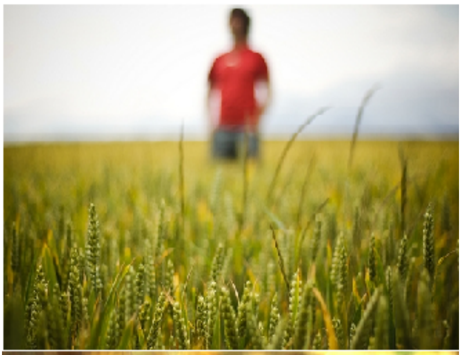
UNIVERSITY OF LEICESTER

The Department of Mathematics, Schools Newsletter, No.4



What has mathematics got to do with a camera's auto focus mechanism? In pure mathematics two valued logic reigns: either a statement is true (1) or it is false (0). In 1965 Lofti Zadeh (http://bit.ly/2j59j3J) proposed an infinite valued or Fuzzy Logic. Truth values here would take any value in the range $0 \leq x \leq 1$. Fuzzy logic has many industrial applications; some examples are digital voice recognition, improved fuel-consumption for automobiles and auto focus control in digital cameras. The digital camera uses fuzzy logic to make assumptions on behalf of the owner. Sometimes it focuses on the object closest to the centre of the viewer. Other times it focuses on the object closest to the camera. The margins of error are acceptable for the camera user whose main concern is album pictures.

Calculating the digits of π.

The numerical value of π is an infinite non-recurring decimal number. The first 25 digits are 3.141592653589793238462643 and the first 10,000 digits of the value of π can be found here: http://bit.ly/JpGrgx. There are various methods to compute these digits of π. Reportedly the most powerful algorithm is the Bailey-Borwein-Plouffe formula (http://bit.ly/1iqeK5Y) but here we describe a method which is familiar in A level further mathematics: Maclaurin series. One use of Maclaurin series for π is to find fast converging approximating finite series.
First we consider the function $f(x) = \tan^{-1}(x), -1 \leq x \leq 1$, whose Maclaurin series is

$$x - \frac{x^3}{3} + \frac{x^5}{5} - ..... + (-1)^n \frac{x^{2n+1}}{2n+1} + ...$$

When $x = 1$ we get:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - ..... + \frac{(-1)^n}{2n+1} + ...$$

Unfortunately around 10,000 terms are needed to obtain the value of π accurate to 4 decimal places! However we can make the following re-arrangement of the terms of the above series by adding and subtracting the term $\frac{1}{4k-2}, k \geq 1$, as follows

$$\frac{\pi}{4} = 1 - \frac{1}{2} + \left(\frac{1}{2} - \frac{1}{3} + \frac{1}{6}\right) - \left(\frac{1}{6} - \frac{1}{5} + \frac{1}{10}\right) + \left(\frac{1}{10} - \frac{1}{7} + \frac{1}{14}\right) + ...$$

After simplification this results in

$$\frac{\pi}{4} = \frac{1}{2} + \frac{1}{2^2-1} - \frac{1}{4^2-1} + \frac{1}{6^2-1} - ....$$

Now 100 terms will give the value of π accurate to 4 decimal places. Other similarly constructed re-arrangments of the original series for 0.25π will result is faster converging series. For example the re-arrangment:

$$\frac{\pi}{4} = \frac{4}{4\times1+1^5} - \frac{4}{4\times3+3^5} + \frac{4}{4\times5+5^5} ....$$

(contact mathsor@le.ac.uk for full details) gives a value of π accurate to 10 decimal places after 100 terms.

A 15th century construction of infinite series for π was connected with the circle and this yielded the unique series (http://bit.ly/19Cua2b)

$$\pi = \sqrt{12}\left(1 - \frac{1}{3\times3} + \frac{1}{3^2\times5} - \frac{1}{3^3\times7} + ...\right)$$

Just 28 terms of this series gives a value of π accurate to 14 decimal places. This improved convergence is testimony to the ingenuity of medieval mathematicians!

Contact: D Almeida, mathsor@le.ac.uk

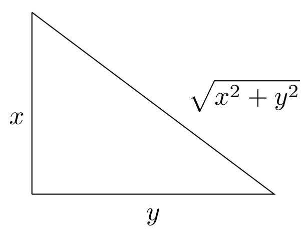
The importance of curiosity in mathematics.

In a lesson on the calculation of the perimeter and area of triangles, a teacher draws two right angled triangles on the board: their dimensions are (6, 8, 10) and (5, 12, 13).

The teacher shows how to calculate their perimeters and areas. The calculations show $P = 24, A = 24$ for the (6, 8, 10) triangle and $P = 30, A = 30$ for the (5, 12, 13) one.

A keen pupil who has realised the importance of patterns in mathematics enquires if $A = P$ for all right angles triangles. The teacher knows that this is not so, but does not want to deflate the keen youngster, so he replies "A good question, Jane! There may be many right angled triangles for which this is true but the only one with whole number sides are the triangles with sides (6, 8, 10) and (5, 12, 13)".

However the teacher has some nagging doubts and wonders to himself if there are any other right angled triangles with integer lengths and $A = P$ that he is not aware of. Once he arrives home in the evening he is implelled to investigate this issue in a wide ranging manner. What follows is the summary of his efforts. Of course the investigation begins by considering a generalised right angled triangle:



If, for this triangle, $P = A$ then:

$\frac{1}{2}xy = x + y + \sqrt{x^2 + y^2}$ or

$xy - 2x - 2y = 2\sqrt{x^2 + y^2}$

Squaring both sides gives

$x^2y^2 + 4x^2 + 4y^2 - 4x^2y + 8xy - 4xy^2 = 4x^2 + 4y^2$

Simplifying this yields:

$x^2y^2 - 4x^2y + 8xy - 4xy^2 = 0$

Or $xy - 4x + 8 - 4y = 0$

Or $y = 4 + \dfrac{8}{x-4}$ ....(*)

This means that y is an integer only if $x - 4$ divides 8. And this can only happen if $x = 12, 8, 6$ or 5. These values only give the right angled triangles (6, 8, 10) or (5, 12, 13).

The teacher feels vindicated but continues the investigation to see what fractional Pythagorean triples might result.

Suppose the triangle above has rational lengths and suppose $x = \dfrac{m}{n}$, where m and n are positive integers.

Then (*) implies $y = \dfrac{4m - 8n}{m - 4n}$ and so

$$x^2 + y^2 = \left(\frac{m}{n}\right)^2 + \left(\frac{4m-8n}{m-4n}\right)^2$$

Some work shows that

$$x^2 + y^2 = \left(\frac{8n^2 - 4mn + m^2}{n(m-4n)}\right)^2$$

Therefore the third side of the triangle above is

$$z = \frac{8n^2 - 4mn + m^2}{n(m-4n)}.$$

This means that providing $m > 4n$, a triangle with sides x, y and z as defined above is right angled with rational lengths and $A = P$.

An implication of the above, the teacher surmises, is that whenever $m > 4n$,
$X = m(m - 4n), Y = n(4m - 8n)$ and $Z = (8n^2 - 4mn + m^2)$

is a Pythagorean triple. He uses a spreadsheet to generate some of these triples. Here is a sample of these fascinating triples presented in columns:

41	84	100	129	480
840	880	1248	920	1768
841	884	1252	929	1832

The teacher feels this investigation has given him some new insights. He feels that the curiosity (and doubts) that impelled him to investigate the pupils has led to some interesting and valuable results. Curiosity is a good thing in mathematics!

An apparently impossible problem.

The triangle below has been divided into four regions. The areas of three of the regions are given. The reader is invited to determine the unknown area of the fourth region, denoted by x. The problem seems impossible but is not. In fact no knowledge beyond GCSE mathematics is required to solve it. Please email your solution to mathsor@le.ac.uk. Solutions will be acknowledged.