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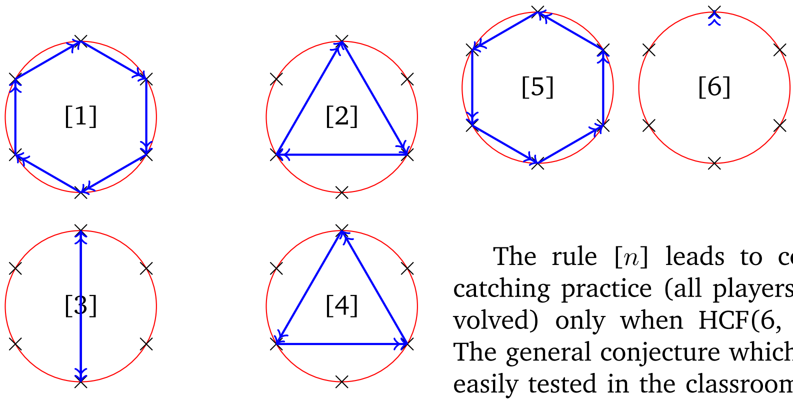
From the ashes of the Houses the rise of the number system!

The scene opposite was painted in 1835 by John Turner and it depicts the houses of parliament burning in 1834 (an event which Turner witnessed). The fire was the direct result of the delay in adopting the Indo-Arabic decimal number system (the number system we use now). in civic life. Conservative elements dictated that these accounts be kept on notches on tally sticks instead. When the Indo-Arabic number system was finally adopted there were a very large number of sticks to be disposed of by burning. The burning of these tally sticks directly resulted in the fire which engulfed both and destroyed both the House of Parliament and the House of Lords.

Charles Dickens (in a speech to the Administrative Reform Association, 27 June 1855) commented on the event thus: "Ages ago a savage mode of keeping accounts on notched sticks was introduced into the Court of Exchequer ....much as Robinson Crusoe kept his calendar on the desert island ..... it took until 1826 to get these sticks abolished. .... In 1834 there was a considerable accumulation of them. The sticks were housed in Westminster.....and so the order went out that they should be privately and confidentially burned. It came to pass that they were burned in a stove in the House of Lords. The stove, overgorged with these preposterous sticks, set fire to the panelling, the panelling set fire to the House of Commons, the two Houses were reduced to ashes .... I think we may reasonably remark, in conclusion, that all obstinate adherence to rubbish which time has long outlived, is certain to have in the soul of it more or less that is pernicious and destructive".

Highest common factors and catching practice.

Six players stand in a circle. One player throws a ball to another player who catches it (indicated by a double headed arrow). There are 6 modes of throwing clockwise: throw to the next player [1], throw to the next player but one [2], throw to the next player but two [3], throw to the next player but three [4], throw to the next player but four [5] and throw to the next player but five [6]. Here are the results of the rules:



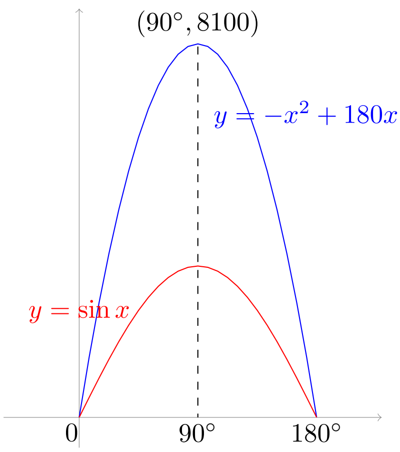
The rule $[n]$ leads to complete catching practice (all players are involved) only when HCF(6, n)= 1. The general conjecture which can be easily tested in the classroom is: For any number N of players the rule $[n]$ leads to complete catching practice only when HCF(N, n)= 1. [1]

[1] The idea for this piece was taken from R P Burn and A Chetwynd, *A cascade of numbers: an introduction to number theory*, 2009.

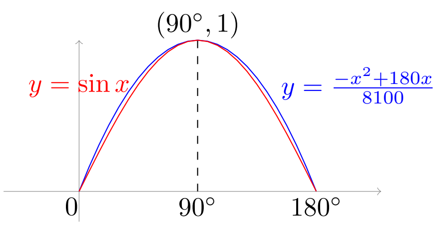
Contact: D Almeida, mathsor@le.ac.uk

Manipulating quadratic functions to *fit* trigonometric functions.

The quadratic function has some interesting properties which are studied in AS mathematics. What may be not known is that there are rational combinations of quadratic functions that approximate trigonometric functions remarkably well. We begin by noting that the sine function *looks* like $f(x) = x(180 - x)$ in the domain $(0°, 180°)$ but it is *taller* than the sine function as this diagram shows:



As is evident $f(x) = x(180 - x)$ is 8100 times taller than the sine function in this domain. If we vertically contract $f(x) = x(180-x)$ by a factor of 8100 we get reasonable fit:



A table of values for the two functions are given below:

$x =$	$\sin x =$	$\frac{-x^2+180x}{8100} =$
0°	0	0
30°	0.5	0.555555
45°	0.707107	0.75
60°	0.866025	0.888888
90°	1	1
120°	0.866025	0.888888
135°	0.707107	0.75
150°	0.5	0.555555
180°	0	0

We see that there are substantial divergences in the table of values for certain angles.

The reason for this is that the vertical scaling factor is not uniform because $\sin 30° = \frac{1}{2}$ while $f(30) = 4500$ which implies a scaling down factor of 9000 while the corresponding values for $x = 90$ imply a scaling down factor of 8100. In addition $\sin 60° = \frac{\sqrt{3}}{2}$ and f(60) = 7200 imply a scaling factor of $\frac{14400}{\sqrt{3}}$.

Thus the scaling function cannot be linear as the difference of scaling down factors is -900 between $x = 30$ and $x = 90$, an average gradient of $\frac{-900}{60} = -15$, while the difference pf scaling down factors is approximately -686 between $x = 60$ and $x = 90$, an average gradient of -22.87.

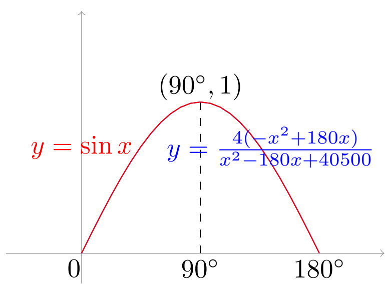
So we let the scaling down function be quadratic and consider the function $S(x) = \dfrac{x(180-x)}{ax^2+bx+c}$, for certain values of a, b and c, to be a good appoximation to the sine function in the domain $(0°, 180°)$.

Now as we know that $\sin 30° = \sin 150°$, we require $S(30) = S(150)$. Some simple algebra then leads to the conclusion that $b = -180a$.

If we then substitute this in the requirement that $S(30) = \sin 30° = \frac{1}{2}$, then it follows that $c = 8100a + 8100$.

Finally if we substitute $b = -180a$ and $c = 8100a + 8100$ into the requirement that $S(60) = \sin 60° = \frac{\sqrt{3}}{2} \approx 0.866$ we find that $a = \frac{8}{0.866} - 9$. This gives $a \approx 0.238$. For the sake of simplicity we take $a = \frac{1}{4}$, which then gives $S(x) = \dfrac{4x(180-x)}{x^2 - 180x + 40500}$. This approximation was originally constructed by Bhaskara I around AD 600: see http://bit.ly/1iuAKz5.

As the diagram below shows $S(x)$ is a remarkably good approximation to the sine function.



The comparative table of values for $y = \sin x$ and $S(x)$ is:

$x =$	$\sin x =$	$\frac{-x^2+180x}{x^2-180x+40500} =$
0°	0	0
10°	0.173648178	0.175257732
20°	0.342020143	0.343163539
30°	0.5	0.5
40°	0.64278761	0.641833811
45°	0.707106781	0.705882353
50°	0.766044443	0.764705882
60°	0.866025404	0.864864865
70°	0.939692621	0.93902439
80°	0.984807753	0.984615385
90°	1	1

As can be seen $S(x)$ produces a table of values closely approximating those of the sine function in the domain $(0°, 90°)$. Further as $S(x)$ has reflection symmetry in $x = 90$, we find that $S(x)$ produces a table of values approximating the sine function in the domain $(0°, 180°)$.

Now using the known fact that $\sin(90 - x)° = \cos x°$ and restricting to the domain $(0°, 90°)$, we naturally find that $C(x) = S(90 - x) = \dfrac{4(8100 - x^2)}{x^2 + 32400}$ is a good approximation to the cosine function.

$x =$	$\cos x =$	$\frac{4(8100-x^2)}{x^2+32400} =$
0°	1	1
10°	0.984807753	0.984615385
20°	0.939692621	0.93902439
30°	0.866025404	0.864864865
40°	0.766044443	0.764705882
45°	0.707106781	0.705882353
50°	0.64278761	0.641833811
60°	0.5	0.5
70°	0.342020143	0.343163539
80°	0.173648178	0.175257732
90°	0	0

Finally $T(x) = \dfrac{S(x)}{C(x)}$, namely

$$T(x) = \frac{x(180-x)(x^2+32400)}{(8100-x^2)(x^2-180x+40500)}$$

will be a good approximation to the tangent function in the the domain $(0°, 90°)$. The reader is invited to explore how good this approximation is.

Contact: D Almeida, mathsor@le.ac.uk