University of Leicester

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The mathematics of realistic paintings.

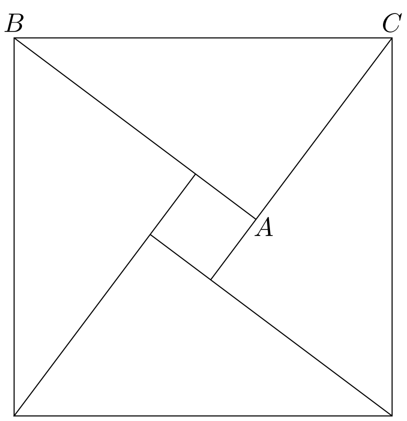
The title of the first painting opposite is *The Miracle of the child falling from the balcony* and was painted by the Italian artist Simone Martini circa 1328. The painting certainly has some artisic merit, but something is amiss. After some reflection one might come to the conclusion that it somehow does not look realistic.
The second painting is aslo by an Italian artist, Raffaello Sanzio da Urbino (commonly known as Raphael). It is entitled *The School of Athens* and was painted in 1518. This painting certainly looks realistic.
But is there a mathematical reason why the first painting is *unrealistic* while the second is *realistic*? The answer is affirmative and it was first provided by the Italian architect Filippo Brunelleschi (1377-1446). Brunelleschi discovered the theory of *perspective*, wherein parallel lines on a horizontal plane depicted in the vertical plane meet at a (vanishing) point. For a picture to be realistic it must have perspective: all sets of parallel lines in the picture must meet at one point. When you view a long straight road it appears to taper to point. Trees nearer you appear to be larger than the ones further away. This is perspective.

- If you identify sets of parallel lines in Martini's painting you will see they do not meet at one point: in fact they meet in several points.
- On the other hand, in Raphael's painting, all parallel lines meet in the centre of the doorway in the background.

It is not surprising to recognise that Martini's painting was made before Brunelleschi discovered his theory of perspective.

Two medieval proofs of 'Pythagoras' theorem'.

These interesting proofs were first presented in the 11^{th} century [1]. The first proof is both visual and algebraic.



Four copies of the right angled triangle ABC with hypotenuse BC are placed with the shorter 'leg' resting on the longer to form a square of side BC and an inner square as shown.

Now the area of each of the four congruent right angled triangles is equal to $\frac{1}{2}AB.AC$.

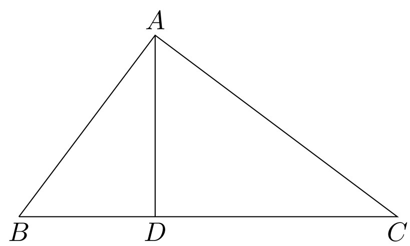
And the area of the square of length BC is equal to the area of the smaller inner square plus the area of the four congruent triangles.

It can be seen that the smaller inner square has length $(AC - AB)$.

So $BC^2 = (AC - AB)^2 + 4.\frac{1}{2}AB.AC$

Simplifying gives $BC^2 = AC^2 + AB^2$.

The second proof requires some angle geometry and algebra. In a right angled triangle ABC with hypotenuse BC, the perpendicular is dropped from A to the hypotenuse BC meeting it at D.



If we let $\angle ABC = \beta$ then, from right angled triangle ADB, we see that $\angle BAD = \frac{\pi}{2} - \beta$.

[1] by the mathematician Bhaskara II in his 11th century study the *Bijaganita*.

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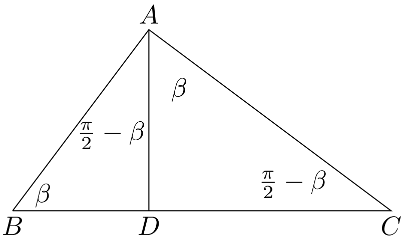
Considering angles in right angled triangle ABC we see that $\angle ACB = \frac{\pi}{2} - \beta$.

Finally considering angles in right angled triangle ADC we see that $\angle DAC = \beta$.

So the picture with these facts is:



This means triangles ABC and ADB are similar. So $\frac{AB}{BC} = \frac{BD}{AB}$.

Thus $BD = \frac{AB^2}{BC}$ ...(1)

Similarly triangles ABC and ADC are similar. So $\frac{AC}{BC} = \frac{DC}{AC}$.

Thus $DC = \frac{AC^2}{BC}$ ...(2)

Using (1) and (2) we have:

$$\begin{aligned} BC &= BD + DC \\ &= \frac{AB^2}{BC} + \frac{AC^2}{BC} \\ &= \frac{AB^2 + AC^2}{BC} \end{aligned}$$

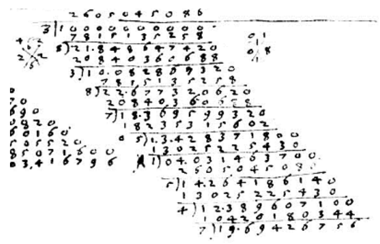
From this we see that $BC^2 = AC^2 + AB^2$.

The medieval calculator.

Christopher Grienberger, who was a mathematics professor at the Jesuit Collegio Romano in Rome in the 16^{th} century, was involved in a race to produce, using manual methods naturally, the most accurate trigonometric tables graduated in increments of 1 minute. Given that there are 60 minutes in one degree, his sine table alone had $60 \times 90 = 5400$ entries!

To get an idea of length of time taken by Grienberger to manually compute his sine table, which had a minimum of 18 places of decimals for each entry, we can refer to his statement in a letter dated 15 Dec 1596: "I have reduced the calculation so that I am able to complete 20 sines sufficiently well, nay, even 30 sines and if school and private reading were not a hindrance....."[2]

Grienberger was a *calculator*, a human involved in the performing complicated arithmetic. The sine table effort occupied Grienberger 'morning, noon and night every day' for a period of 3 years. He maintained accuracy in arithmetic by using the method of 'casting out 9's' [3]. Here is a sample of his arithmetic:



Of course, now a calculator is an inanimate object that can effortlessly and quickly perform the complicated arithmetic that Grienberger had to laboriously do. Nevertheless it is valuable to know about Grienberger's methods as it informs and illuminates sixth form trigonometry.

Grienberger's sine table was constructed with a *seed* value of sin 1' (1' = 1 minute) to 22 places of decimals. His astounding manually computed value for sin1' was 0.0002908882045634245911, which is an amazingly accurate estimate given that the correct value is 0.0002908882045634245964 to 22 places of decimals.

After having the value of sine 1', Grienberger computed the value of cosine 1' using the standard identity $\cos x = \sqrt{1 - \sin^2 x}$. Of course, there was the *small* matter of squaring a number with 22 places of decimals and square rooting a number with even more decimal places.

Grienberger's table was then built up from the initial values of sine 1' and cosine 1' using formulae which are well known in sixth form mathematics.

1) To get sine and cosine of $2'$ he used double angle formulae:

$\sin 2' = 2 \sin 1' \cos 1'$

$\cos 2' = 1 - 2\sin^2 1'$

2) To get sines and cosines of subsequent minutes he used these identities which the reader is invited to verify:

$\sin A'$
$\equiv 2 \sin 1' \cos(A-1)' + \sin(A-2)'$.

$\cos A'$
$\equiv \cos(A-2)' - 2 \sin 1' \sin(A-1)'$.

So, for example, having the values of sine and cosine of 1' and 2' he could compute:

$\sin 3' = 2 \sin 1' \cos 2' + \sin 1'$
$\cos 3' = \cos 1' - 2 \sin 1' \sin 2'$

And the procedure was repeated for sines and cosines of subsequent minutes.

3) To obtain sines of bigger angles Grienberger used these formulae:

$\sin(60° + A°) \equiv \sin A° + \sin(60° - A°)$
$\sin(90° - A°) \equiv \cos A°$

So, for example, having the values of sines of $2°14'$, $3°28'$, $57°46'$, and $56°32'$ he could compute:

$\sin 62°14' = \sin 2°14' + \sin 57°46'$
$\sin 63°28' = \sin 3°28' + \sin 56°32'$.

And knowing the value of cosine of $\theta° \leq 45°$ he could compute:

$\sin 54°34' = \cos 35°26'$
$\sin 74°4' = \cos 15°56'$, etc.

A sample from from Grienberger's original manuscript is presented below. Here you should be able to identify Grienberger's sine values for the angles $2°14', 2°16', 87°46'$ and $87°44'$.

Grienberger completed his sine table on 14 December 1596 and the completed manuscript, GES874, is lodged in the Bibilioteca Nazionale in Rome. Due to certain technical difficulties involving tangent values his tables were finally published in abridged form in 1630.

Readers wishing to find out more about the history and construction of trigonometric tables in medeival times can refer to http://bit.ly/ZmBsmT.

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[2]Thanks to Dr R P Burn, University of Exeter, for the translation from the Latin.
[3]See http://bit.ly/wUkTA to find out about 'casting out 9's'.

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