The Department of Mathematics, Newsletter to Schools, 2015/16, No.1

Algebra and broken bones.

Al-Khwarizmi, a mathematician from Persia (now Iran), wrote the first treatise on algebra: Hisab al-jabr w'al-muqabala in 820 AD. The word algebra is a corruption of *al-jabr* which means restoration. Now if something needs restoring it must have been damaged, broken or fractured. How is this possible in algebra? Here, one possible way of seeing this, is as follows: when we open brackets we fracture and when we factorise we restore.

The restoration of broken parts meaning of *al-jabr* is still evident in modern times. In Spain, where the Arabs held sway for a long period, *al-jabr* was also taken to mean the restoration of broken bones. So there arose the profession of *algebrista* who healed fractured bones. The algebrista also practiced medical blood letting. So in times past one would find the trade sign Algebrista y Sangrador in many Spanish cities. We can still find evidence of this profession in the definition of algebra in current Spanish dictionaries. The following definition of the word 'algebra' is from the Real Academia Espanola (http://buscon.rae.es/drae/srv/search?id=BZWT11t5vDXX2Smmq8pf):

1. f. Parte de las matematicas en la cual las operaciones aritmeticas son generalizadas empleando numeros, letras y signos. (*Translation: That part of mathematics in which arithmetic operations are generlaized using numbers, letters and symbols*.)
2. f. desus. Arte de restituir a su lugar los huesos dislocados (*Translation: the art of restoring broken bones to their original locations*.)

Moessner's theorem or how to generate powers of numbers in an interesting way.

There is an interesting way to obtain powers of integers by cumulative addition of specific sequences of numbers. It represents a good opportunity to get pupils to see some of the rich patterns in natural numbers.

For n^{th} powers the procedure involves the removal of every n^{th} number from the list of natural numbers.

A. Square numbers.
Start with the natural numbers:

$1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, \ldots.$

First remove every 2^{nd} number from this list:
$1, \quad 3, \quad 5, \quad 7, \quad 9, \quad 11, \quad 13, \ldots.$
Now add cumulatively from the left:
$1, \quad 3, \quad 5, \quad 7, \quad 9, \quad 11, \quad 13, \ldots.$
1, 4, 9, 16, 25, 36, 49,....

Which is a list of square numbers.

B. Cubes.
Start with the natural numbers:
$1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, \ldots.$
Now remove every 3^{rd} number:
$1, 2, \quad 4, 5, \quad 7, 8, \quad 10, 11, \quad 13, \ldots.$
Now add cumulatively from the left:
$1, 2, \quad 4, 5, \quad 7, 8, \quad 10, 11, \quad 13, \ldots.$
$1, 3, \quad 7, 12, \quad 19, 27, \quad 37, 48, \quad 61, \ldots.$
Remove every 2^{nd} number :
$1, \quad 7, \quad 19, \quad 37, \quad 61, \ldots.$
Add cumulatively from the left:
1, 8, 27, 64, 125,....

And we have the list of cubes.

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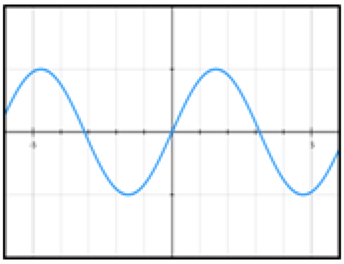
For *fourth powers*:

Delete every 4^{th} number $\rightarrow$
do cumulative addition $\rightarrow$
delete every 3^{rd} number $\rightarrow$
do cumulative addition $\rightarrow$
delete every 2^{nd} number $\rightarrow$
do cumulative addition to obtain fourth powers.

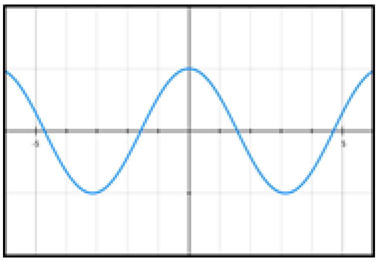
And so on for higher powers. This highly interesting procedure arises from Moessner's theorem. For more about this theorem please do see: http://bit.ly/18IpPVL.

The odd thing about odd and even functions.

A function $f(x)$, with a domain symmetrical about the origin, that satisfies that $f(-x) = -f(x)$ for all x in its domain is an odd function. An odd function has a graph that possesses half turn rotational symmetry about the origin.
$f(x) = \sin x$ is odd as illustrated by its the graph:



A function $g(x)$, with a domain symmetrical about the origin, that satisfies that $g(-x) = g(x)$ for all x in its domain is an even function. An even function has a graph with reflection symmetry about the y - axis.
$g(x) = \cos x$ is even as illustrated by its the graph:

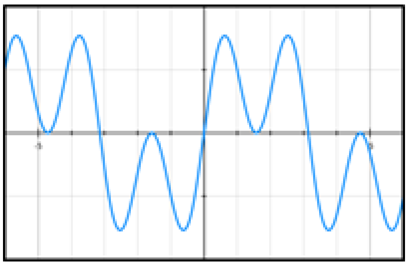


It can easily be shown that, for any non-zero integer k, $f_k(x) = \sin kx$ is an odd function and $g_k(x) = \cos kx$ is an even function.

The questions that now arise from our knowledge of odd and even numbers are:

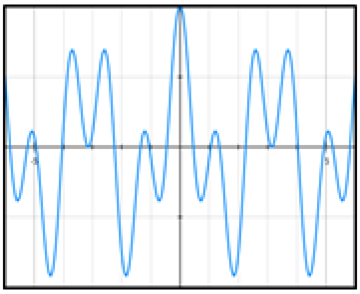
1. Is the sum of two odd functions even?
2. Is the sum of two even functions even?

Consider $f(x) = f_1(x) + f_3(x) = \sin x + \sin 3x$. Its graph is:



This seems to imply that the sum of two odd functions is odd (unlike the case with numbers).
The following argument proves this conjecture:
Let F and F' be odd functions with, for the sake of simplicity, identical domains. Then
$(F + F')(-x) = F(-x) + F'(-x)$
$= -F(x) - F'(x)$: F and F' are odd
$= -(F + F')(x)$
Therefore $(F + F')$ is odd.

Now Consider $g(x) = g_2(x) + g_5(x) = \cos 2x + \cos 5x$. Its graph is:

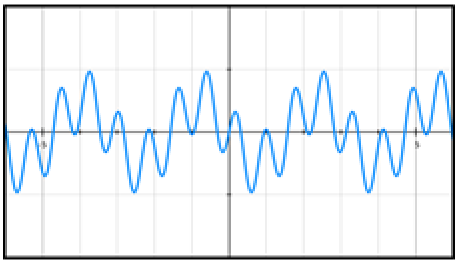


The evidence seems to suggest that the sum of two even functions is even (like the case with numbers).
The following argument proves the conjecture:
Let G and G' be even functions.
Then:
$(G + G')(-x) = G(-x) + G'(-x)$
$= G(x) + G'(x)$: G and G' are even
$= (G + G')(x)$
Thus $(G + G')$ is even.

So the answers to the two questions are:

1. Is the sum of two odd functions even? **No. It is still odd.**
2. Is the sum of two even functions even? **Yes.**

Nothing, of course, can be said of the sum of an odd and an even function (examine the graph of a function that is the sum of an odd and an even function. Also see exercise* below). But what about the product of odd and even function? Let's examine the graph of this product of an odd and an even function:
$f_3(x)g_5(x) = \sin 3x \cos 5x$.



Its graph above implies that $f_3(x)g_5(x)$ is odd. The result that the product of an odd function F and an even function G is odd, unlike with numbers, and this result true in general. Here is the proof:
$FG(-x) = F(-x)G(-x)$
$= -F(x).G(x) = -FG(x)$
So FG is odd.

The reader is invited to investigate the odd arithmetic of odd and even functions further. The reader is also invited to verify that *every function $h(x)$ with the requisite domain is the sum of an odd function
$F(x) = \frac{1}{2}(h(x) - h(-x))$
and an even function
$G(x) = \frac{1}{2}(h(x) + h(-x))$.

This article is based on interactions with students in FMSP sixth form courses. More on this topic here: http://bit.ly/15i0SjU

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