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$$\sqrt{144} = \pm 12$$

$$\frac{a}{b} \div \frac{c}{d} = \frac{a \div c}{b \div d}$$

Misconceptions about misconceptions in mathematics.

We all know that younger pupils learning mathematics may tend to adopt incorrect algorithms. For example, a pupil may add the fractions $\frac{2}{3} + \frac{1}{4}$ by saying the result is $\frac{2+1}{3+4} = \frac{3}{7}$.
However it can also happen that misconceptions appear also in secondary mathematics textbooks. The first example in the image opposite is taken from a secondary school text. And it is incorrect mathematics for the simple reason that the $\sqrt{}$ symbol refers to the positive square root of a positive real number. So $\sqrt{144} = +12$. This misconception possibly arises by thinking that $\sqrt{}$ comes from inverting the equality $(\pm a)^2 = a^2$. Anecdotal evidence suggests that this misconception, that $\sqrt{a^2} = \pm a$, persists in the minds of a substantial proportion of older learners.
In another case adult learners were asked if the algorithm in the second part of the image, namely that $\frac{a}{b} \div \frac{c}{d} = \frac{a \div c}{b \div d}$, was true or false. The majority of respondents, possibly remembering that $\frac{a}{b} + \frac{c}{d} \neq \frac{a+c}{b+d}$, stated that it was false. It is, in fact, a correct algorithm but not adopted in the arithmetic for the simple reason that it does not always give the answer as conventional rational numbers. For example $\frac{4}{9} \div \frac{2}{3} = \frac{4 \div 2}{9 \div 3} = \frac{2}{3}$ is ok. But $\frac{4}{9} \div \frac{3}{4} = \frac{4 \div 3}{9 \div 4}$ gives $\frac{1.33..}{2.25}$, which is correct but awkward because it is not a conventional fraction. We would rather have $\frac{16}{27}$.

A rational explanation why there are irrational numbers.

The rational numbers are just the integers and fractions. The former have physical representation when counting objects. The latter can be represented when we divide or share an integer n by another integer m.

The rational numbers have specific types of decimal representation. They have either terminating or recurring decimal representations.

An **integer** n can be represented as just $n.0$, for example $7 = 7.0$.

Fractions have either terminating or recurring decimal representations. For example $\frac{2}{5} = 0.4$ and $\frac{2}{3} = 0.\overline{6}$.
To prove that fractions have either terminating or recurring decimal representations we first state that fractions are either of the form $\frac{A}{2^m 5^n}$, where m and n are positive integers, or of the form $\frac{A}{B}$, where B is not of the form $2^m 5^n$.

In the **former** case $\frac{A}{2^m 5^n} = \frac{A \times 5^{m-n}}{10^m}$ or $\frac{A}{2^m 5^n} = \frac{A \times 2^{n-m}}{10^n}$, depending on which of m or n is larger: in both cases it is easy to see that we have a terminating decimal.

In the **latter** case where we have fractions of the type $\frac{A}{B}$, where B is not of the form $2^m 5^n$, let us assume that $B > A$. When we divide A by B there are at most $B - 1$ remainders, and these remainders will repeat at some point. As a concrete example when we divide 3 by 7 the possible remainders in the long division are $1, 2, 3, 4, 5$ or 6 (it cannot be 0 because then the fraction would be terminating and so B would be of the form $2^m 5^n$). So at some point in the long division process a remainder must recur. In fact the **6** unique remainders when we divide 3 by 7 are 2,6,4,5,1,3 with respective quotients 4,2,8,5,7,1 and this pattern is repeated. So $\frac{3}{7}$ has a decimal representation with **6** recurring digits: $0.\overline{428571}$. Sometimes the recurring digits occur later than the first place; for example $\frac{5}{12} = 0.42\overline{6}$. And using secondary school algebra we can also find the fraction equal to any number whose decimal representation has recurring decimal places.

So a number with a decimal representation that is neither terminating or recurring, such as $0.1010010001000001......$, is not rational. These are the irrational numbers, but unlike rational numbers a physical analogy for them is not easy.

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Surprising results from sums of sequences of natural numbers.

We first need to note that the sum of an arithmetic sequence $a, a+d, a+2d, ..., a+(n-1)d$ is well known. It is $S_n = \frac{1}{2}n(2a+(n-1)d)$.

Now consider the sequence of natural numbers $1, 2, 3, 4, ..., n, ...$. If we sum cumulatively from the left we get the sequence:

$$1, 3, 6, 10, ...$$

which appear to be the familiar **triangle** numbers. To prove that this is always the case we note that the first n natural numbers form an arithmetic sequence with $a = d = 1$ so its sum is: $S_n = \frac{1}{2}n(2+(n-1)) = \frac{1}{2}n(n+1)$. And this is the formula for triangle numbers.

Next we form the sequence of the natural numbers with the **even** numbered terms removed:

$$1, 3, 5, 7, ..., (2n-1), ...$$

And then sum cumulatively from the left to get:

$$1, 4, 9, 16, ...$$

To be sure we always get **square** numbers we need to sum the n^{th} cumulative sum. Here $a = 1$ and $d = 2$, so the sum is

$$S_n = \frac{1}{2}n(2+(n-1)2) = n^2.$$

So the cumulative sums are indeed always **square** numbers.

Now we form the sequence first n natural numbers with **multiples of 3** removed, find cumulative sums, remove **even** numbered terms, and then cumulatively sum from the left. The procedure in steps is:

$$1, 2, 4, 5, 7, 8, ..., (3n+1), (3n+2), ...$$

The cumulative sums are:

$$1, 3, 7, 12, 19, 27, ...$$

Removing even numbered terms gives:

$$1, 7, 19, ...$$

Finally summing cumulatively from the left gives:

$$1, 8, 27, ...$$

The sums appear to be **cubed** numbers. Since mathematics is about making sure we will prove this by using the arithmetic series sum formula and the formula for the sum SQ of the first n squares, namely,

$$SQ = \frac{1}{6}n(n+1)(2n+1).....(*)$$

We first go back to the finite sequence

$$1, 2, 4, 5, 7, 8, ..., (3n+1), (3n+2)$$

And find the cumulative sequence S up to $(3n+2)$. The cumulative sum is the sum of the natural numbers from 1 to $(3n+2)$ less the positive multiples of 3 up to $3n$. So

$$S = \frac{1}{2}(3n+2)(3n+3) - 3 \times \frac{1}{2}n(n+1).$$

Or $S = \frac{3}{2}(n+1)(3n+2-n)$.

Or $S = \frac{3}{2}(n+1)(2n+2)$.

That is, $S = 3(n+1)^2$.

Thus the generalised cumulative sums after the multiples of 3 are removed is

$$1, 3, 7, 12, 19, 27, ...$$
$$...., 3(n+1)^2 - (3n+2), 3(n+1)^2, ...$$

The even numbered terms are 3 times square numbers are removed to form the next sequence. The terms that are left are seen to have the general form:

$$3(n+1)^2 - (3n+2)$$

So the final sequence to be cumulatively summed is easily seen to be:

$$(3 \times 1^2 - 3 \times 0 + 2) = 1,$$
$$(3 \times 2^2 - 3 \times 1 + 2) = 7,$$
$$(3 \times 3^2 - 3 \times 2 + 2) = 19,$$
.
.
$$3(n+1)^2 - (3n+2),$$
.
.

We will find the sum T up to the $(n+1)^{\text{st}}$ term $3(n+1)^2 - (3n+2)$ shown above.

We can see that T comprises of 3 times the sum of the first $(n+1)$ square numbers minus 3 times the sum of the first n natural numbers minus $2(n+1)$.

Using the arithmetic series formula and formula (*) we get:

$$T = 3 \times \frac{1}{6}(n+1)(n+2)(2n+3)$$
$$-3 \times \frac{1}{2}n(n+1) - 2(n+1).$$

Factoring out $\frac{1}{2}(n+1)$ we then get

$$T = \frac{1}{2}(n+1)\left((n+2)(2n+3) - 3n - 4\right).$$

Or

$$T = \frac{1}{2}(n+1)\left(2n^2 + 7n + 6 - 3n - 4\right).$$

Or

$$T = \frac{1}{2}(n+1)\left(2n^2 + 4n + 2\right).$$

Or

$$T = (n+1)\left(n^2 + 2n + 1\right).$$

Hence $T = (n+1)^3$. And this proves that the final cumulative sums are indeed **cubed** numbers.

The arguments thus far are confirmation of the first two conjectures made about Moessner sequences in newsletter 1.

To generate **fourth** powers using the Moessner procedure one has to:

1. Remove every 4th number from the sequence of natural numbers.
2. Do cumulative addition.
3. Remove every 3rd number.
4. Do cumulative addition.
5. Remove every 2nd number.
6. Do cumulative addition to obtain fourth powers.

The reader is encouraged to prove the **fourth** power conjecture by extending the argument.

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