Year13 challenge problems.

These problems are more challenging than those seen in Advanced level mathematics papers. Their focus is on mathematical thinking and, as a consequence, on STEP preparation. As such they should appeal to students who like and do mathematics for its own sake and also to those students who intend to sit STEP examinations.
The problems are from various sources both published and unpublished. The degree of difficulty of the problems are based on my own experiences solving the problems and are rated A (easier) to E (hardest). The first 10 problems are A, A/B or B/C, thereafter the problems are rated A/B to E in no particular order. For feedback on solutions and/or a standard solution email mathsor@le.ac.uk

1. (A) A triangle has sides of length a, b and c which satisfy the relation $a^2 + b^2 + c^2 - ab - bc - ca = 0$. Prove that the triangle is equilateral.

2. (A/B) The Arithmetic Mean-Geometric Mean inequality states that if $x_1, x_2, ..., x_n$ are n non-negative numbers then
$$\frac{x_1 + x_2 + ... + x_n}{n} \geq \sqrt[n]{x_1 x_2 .....x_n}.$$
If $x > 0$ then use the Arithmetic Mean-Geometric Mean inequality to prove that the greatest possible value of $\dfrac{x^n}{1 + x + x^2 + ... + x^{2n}}$ is $\dfrac{1}{2n+1}$.

3. (A/B) Given that A, B and C are angles in a triangle. Prove that:
$$\tan\frac{A}{2}\tan\frac{B}{2} + \tan\frac{B}{2}\tan\frac{C}{2} + \tan\frac{C}{2}\tan\frac{A}{2} = 1$$

4. (A/B) Given that a, b and c are distinct real numbers and that p, q and r are some real numbers such that $p\mathrm{e}^{ax} + q\mathrm{e}^{bx} + r\mathrm{e}^{cx} = 0$ for all values of x. Prove that $p = q = r = 0$.

5. (A/B) Given that $\mathrm{f}(x) = (1 - kx)(1 - k^2x)......(1 - k^{n-1}x)$, where k is a real number, in expanded form is $\mathrm{f}(x) = 1 + a_1x + a_2x^2 + ... + a_{n-1}x^{n-1}$.
   (a) Prove that $(1 - k^n x)\mathrm{f}(x) = (1 - kx)\mathrm{f}(kx)$.
   (b) Prove that $a_r = \dfrac{(k^n - k)(k^n - k^2)......(k^n - k^r)}{(1 - k)(1 - k^2).....(1 - k^r)}$.

6. (A/B: if correct strategy is used). Solve the equation

$$\sqrt{x+3-4\sqrt{x-1}}+\sqrt{x+8-6\sqrt{x-1}}=k,$$

where k is some positive real number.

7. (B/C) The quartic equation $x^4-10x^3+24x^2+4x-4=0$ has roots a, b, c and d such that $ab+cd=0$. Find the values of a, b, c and d.

8. (B/C)

   (a) Solve the equation $1-x+\frac{x(x-1)}{2!}=0$.

   (b) Solve the equation $1-x+\frac{x(x-1)}{2!}-\frac{x(x-1)(x-2)}{3!}=0$.

   (c) Make a conjecture about the roots of the equation

   $$1-x+\frac{x(x-1)}{2!}-\frac{x(x-1)(x-2)}{3!}+.......+(-1)^n\frac{x(x-1)(x-2).....(x-n+1)}{n!}=0$$

   Prove the conjecture using mathematical induction.

9. (B/C) Find all roots of the equation $x^3=(4-x^2)^{\frac{3}{2}}$. Write all complex roots in the form $a+bi$, where a and b are real.

10. (B/C) This problem is trivial if the Arithmetic Mean-Geometric Mean inequality is used. In order for it to be a challenge the required proof should only use algebra.

    Given that a_1, a_2, a_3 and a_4 are positive numbers.

    Prove that $\sum_{k=1}^{r} a_k \sum_{k=1}^{r} \frac{1}{a_k} \geq r^2$, for $1 \leq r \leq 4$.

11. (C/D) (This problem is due to a Belgian artist Patrick Shimmel)

    Given that
    $$f(x,y)=x^2+2x(y+1)+y^2+y+1$$
    and
    $$g(x,y)=x^2+2x(y+1)+y^2+3y+2$$
    where x and y are non-negative integers. Prove that the range of $f(x,y)$ is disjoint from the range of $g(x,y)$ and that the union of the ranges is the entire set of natural numbers.

12. (C/D) Solve the set of simultaneous equations

$$x^3 + y^3 + z^3 = 8 \quad (1)$$
$$x^2 + y^2 + z^2 = 4 \quad (2)$$
$$x + y + z = 2 \quad (3)$$

13. (D/E)

(a) Show that $x = -y - z$ satisfies the equation $x^3 + y^3 + z^3 - 3xyz = 0$. Hence express $(x^3 + y^3 + z^3 - 3xyz)$ as a product of two factors.

(b) Solve the set of simultaneous equations given that $x, y, z > 0$.

$$x^3 + y^3 + z^3 - 3xyz = 27 \quad (4)$$
$$x^2 + y^2 - z^2 = 9 \quad (5)$$
$$x + y + z = 6 \quad (6)$$

14. (D/E)

(a) Evaluate $\sum_{r=1}^{n} \cos rx$ and $\sum_{r=1}^{n} \sin rx$.

(b) Deduce that $\frac{n+1}{2} + \sum_{r=1}^{n} (n - r + 1) \cos rx = \frac{1}{2} \frac{\sin^2\left(\frac{n+1}{2}\right) x}{\sin^2\left(\frac{x}{2}\right)}$

15. (E) Express each numerator and denominator below as a sum of an integer and rational number. By considering the general form of the numerators and denominators, or otherwise, find the exact value the product.

$$\frac{\frac{65}{4}}{\frac{5}{4}} \times \frac{\frac{1025}{4}}{\frac{325}{4}} \times \frac{\frac{5185}{4}}{\frac{2501}{4}} \times \text{........} \times \frac{\frac{400000001}{4}}{\frac{384238405}{4}}$$

16. (B/C)

(a) By examining the values of $2n^4 + 2(n + 1)^4$ for $n = 0, 1, 2, 3$ form a conjecture about the values of $2n^4 + 2(n + 1)^4 + 2$ whenever n is an integer. Prove the conjecture.

(b) By considering the equation $x^4 + (x+1)^4 = y^2 + (y+1)^2$ as a quadratic equation in y show that there are a finite number of integer solutions of the equation $x^4 + (x + 1)^4 = y^2 + (y + 1)^2$.

17. (B) The roots of a cubic equation $f(x) = 0$ are a, b and c. The roots of the equation $f'(x) = 0$ are p and q. Prove the following:

$$(a - p)(a - q) = \frac{1}{3}(a - b)(a - c) \quad (7)$$
$$\left(\frac{b + c}{2} - p\right)\left(\frac{b + c}{2} - q\right) = -\frac{1}{12}(b - c)^2 \quad (8)$$

18. (C/D)

    (a) By considering the global maximum of the function $y = \ln x - x + 1$ prove that $\ln x \leq x - 1$ whenever $x > 0$.

    (b) Let x_i and t_i, $1 \leq i \leq n$, be positive numbers such that

    $$x_1 t_1 + x_2 t_2 + ... + x_n t_n = t_1 + t_2 + ... + t_n = 1$$

    Prove that

    $$x_1^{t_1} x_2^{t_2} ...... x_n^{t_n} \leq 1$$

    (c) Deduce that if a_i and $p_i, 1 \leq i \leq n$, are positive numbers then

    $$a_1^{p_1} a_2^{p_2} .... a_n^{p_n} \leq \left(\frac{a_1 p_1 + a_2 p_2 + ... + a_n p_n}{p_1 + p_2 + ... + p_n} \right)^{p_1 + p_2 + ... + p_n}$$

19. (C/D) A sequence of real numbers $a_n, n \geq 0$, satisfies

    $$a_n - 2a_{n+1} + a_{n+2} \sin^2 \alpha = 0, \;\; n \geq 2$$

    where
    $a_0 = 2\cos\alpha$ and $a_1 = \dfrac{2\cos\alpha}{\sin^2\alpha}$, $\cos\alpha \neq \pm 1$.

    (a) Prove that $a_r = \dfrac{\cos\alpha}{(1+\cos\alpha)^r} + \dfrac{\cos\alpha}{(1-\cos\alpha)^r}, r \geq 0$

    (b) Prove that $\sum\limits_{r=1}^{n} a_r = \dfrac{1}{(1-\cos\alpha)^n} - \dfrac{1}{(1+\cos\alpha)^n}$.

    (c) If $t = \tan\left(\frac{1}{2}\alpha\right)$ then show that $\dfrac{1}{1-\cos\alpha} = \dfrac{1+t^2}{2t^2}$;
    $\dfrac{1}{1\cos\alpha} = \dfrac{1+t^2}{2}$; $\cos\alpha = \dfrac{1-t^2}{1+t^2}$.

    By using these subsitutions in the sum in part (b) show that

    $$\sum_{r=1}^{n} (1+t^2)^r (t^{2n-2r} + t^{2n}) 2^{n-r} = (1+t^2)^{n+1} \sum_{r=1}^{n} t^{2r-2}$$

    .

20. (D/E) Given that $a_1, a_2, ..., a_n$ are the n roots of the the equation $x^n + nax - b = 0$. Prove that

    $$(a_1 - a_2)(a_1 - a_3)......(a_1 - a_n) = n(a_1^{n-1} + a)$$

    Show that the equation $y(y+na)^{n-1} = b^{n-1}$ has roots $a_1^{n-1}, a_2^{n-1}, ..., a_n^{n-1}$.

21. (C/D) A sequence of real numbers a_r is defined by

$$a_0 = 0; \quad a_{r+1}\cos rx - a_r\cos(r+1)x = 1; \quad r \geq 0.$$

If $\cos rx \neq 0$ for any r prove that

$$a_{r+2} - 2a_{r+1}\cos x + a_r = 0, \quad r \geq 0.$$

Hence prove by mathematical induction that $a_r = \dfrac{\sin rx}{\sin x}$.

Deduce the value of $\sum\limits_{r=1}^{n} \sec rx \sec(r+1)x$.

22. (C/D)

   (a) Prove that $1 + 2\sum\limits_{r=1}^{n} \cos 2rx = \operatorname{cosec} x \sin(2n+1)x$.

   (b) Prove that $\sum\limits_{r=0}^{2n} \cos(x + rsA) = 0$, where $A = \dfrac{2\pi}{2n+1}$ and s is an integer which is not a multiple of $2n+1$.

   (c) Deduce that $\sum\limits_{s=0}^{2n} \operatorname{cosec}(x + sA) = (2n+1)\operatorname{cosec}(2n+1)x$.

23. (D/E)

   (a) Prove that if the cubic equation $ax^3 + x^2 - 3bx + 3b^2 = 0$ has two non-zero equal roots then it has three equal roots. Determine a in terms of b.

   (b) Prove that if the quartic equation $x^4 + 4ax^3 + 2x^2 - 4bx + 3b^2 = 0$ has three non-zero equal roots then it has four equal roots. Determine the values a and b.

24. (C/D) Solve the equation $x^6 - 6x^5 + 5x^4 + 20x^3 - 14x^2 - 28x - 8 = 0$ given that it is unaltered by replacing x by $2 - x$.

25. (D/E) Given that n is a positive even integer. Use DeMoivre's theorem to show that

$$\cos nx = a_n \cos^n x - a_{n-2}\cos^{n-2} x + a_{n-4}\cos^{n-4} x - \ldots\ldots \pm a_0$$

where the $a_n, a_{n-2}, a_{n-4}, \ldots.., a_0$ are positive integers.

Prove that $a_n + a_{n-2} + a_{n-4} + \ldots.. + a_0 = \dfrac{1}{2}\left((1+\sqrt{2})^n + (1-\sqrt{2})^n\right)$.

26. (D/E) In this problem n is an odd positive integer.

   (a) Use DeMoivre's theorem to express $\cos nx$ as a polynomial in $\cos x$:

$$Q(\cos x) = c_n \cos^n x + c_{n-2}\cos^{n-2} x + \ldots.. + c_1 \cos x$$

   Determine the value of c_n.

(b) Hence find the value of $\prod_{k=1}^{1110} \cos\left(\frac{2k\pi}{1111}\right)$.

27. (C/D)

(a) Use DeMoivre's theorem to express $\sin 5x$ as a polynomial in $\sin x$. Hence show that $\frac{1-\sin 5x}{1-\sin x}$ is the square of a polynomial in $\sin x$.

(b) If n is a positive integer show that $\sin 2nx$ and $\cos(2n+1)x$ can be both expressed as the product of $\cos x$ and some polynomial in $\sin x$. Show that $\frac{1-\sin(4n+1)x}{1-\sin x}$ is the square of a polynomial in $\sin x$.

28. (C/D) Let v be some real number and i the imaginary square root of -1. Then it is given that $\cos(iv) = \cosh v$ and $\sin(iv) = i \sinh v$.

It is also given that the formulae $\cos(A+B) = \cos A \cos B - \sin A \sin B$ and $\sin(A+B) = \sin A \cos B + \cos A \sin B$ are also valid for complex numbers A and B.

Now suppose that $(x+iy) = \cot(u+iv)$ where x, y, u and v are some real numbers. Show that

$$x^2 + y^2 - 2x \cot 2u - 1 = 0 \quad \text{and} \quad x^2 + y^2 + 2y \coth 2v + 1 = 0$$

Using coordinate geometry further show that, for constant u and v, the two circles represented by the equations above cut orthogonally (the tangents at the points of intersection are perpendicular) and that the angle between the lines joining $(0, \pm 1)$ to a point on the first circle is $2u$.

29. (B/C) The sequence of real numbers a_i, $i \geq 0$, with $a_0 = 2$ and $a_1 = -p$ satisfies the recurrence relation $a_{n+2} = -pa_{n+1} - qa_n$. Show that

$$\sum_{r=0}^{\infty} a_r t^r = \frac{2+pt}{1+pt+qt^2}.$$

By squaring $a_{n+2} = -pa_{n+1} - qa_n$, $a_{n+3} = -pa_{n+2} - qa_{n+1}$ and then considering $a_{n+3}^2 + qa_{n+2}^2$, find the recurrence relation, which involves $a_{n+3}^2, a_{n+2}^2, a_{n+1}^2$ and a_n^2, satisfied by the sequence a_i^2, $i \geq 0$. Hence find the sum of the series $\sum_{r=0}^{\infty} a_r^2 t^r$.

30. (a) (A) Use DeMoivre's theorem to prove that $\left(\dfrac{1+i\tan x}{1-i\tan x}\right)^n = \dfrac{1+i\tan nx}{1-i\tan nx}$.

(b) (C/D) By solving the relation in (a) for $\tan nx$ show that

$$\tan nx = \frac{p(\tan x)}{q(\tan x)}$$

where $p(\tan x)$ and $q(\tan x)$ are polynomials in $\tan x$.

(c) (D/E) By re-writing the equation $\cot nx = 1$ as a polynomial in $\cot x$ and using the theory of roots of polynomial equations prove that

$$\sum_{r=0}^{n-1} \cot\left(\frac{(4r+1)\pi}{4n}\right) = n \quad \text{and} \quad \sum_{r=0}^{n-1} \cot^2\left(\frac{(4r+1)\pi}{4n}\right) = n(2n-1).$$

31. (C/D) Given that $\alpha < \beta$ are some real constants. Use the substitution $\mathrm{e}^{\theta} = \mathrm{e}^{\alpha}\cos^2 t + \mathrm{e}^{\beta}\sin^2 t$ to show that

$$I = \int_{\alpha}^{\beta} \frac{1}{\sqrt{(\mathrm{e}^{\theta}-\mathrm{e}^{\alpha})(\mathrm{e}^{\beta}-\mathrm{e}^{\theta})}}\,\mathrm{d}\theta = \pi \mathrm{e}^{-1/2(\beta+\alpha)}$$

32. (B/C) Prove that $I = \int_0^{\pi/3}(1+\tan^6 x)\,\mathrm{d}x = \dfrac{9\sqrt{3}}{5}$

33. (B/C) Prove that $I = \int_0^{\pi/2}\sqrt{\dfrac{1-\sin x}{1+\sin x}}\,\mathrm{d}x = \ln 2$

34. (C/D) Use a trignometric substitution and then half angle formulae to show that

$$\int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} \frac{1}{(\sqrt{3}-\sqrt{2}x)\sqrt{1-x^2}}\,\mathrm{d}x = \frac{\pi}{2}$$

35. (C/D) S is the infinite series given by:

$$S = \cos x - \frac{1}{2}\cos 3x + \frac{1.3}{2.4}\cos 5x - \frac{1.3.5}{2.4.6}\cos 7x + \ldots + (-1)^r \frac{1.3.5...(2r-1)}{2.4.6....(2r)}\cos(2r+1)x + \ldots$$

Use the fact that $e^{i\theta} = \cos\theta + i\sin\theta$ to prove that $S = \dfrac{\cos(x/2)}{\sqrt{2\cos x}}$.

36. (A/B) Use the Arithmetic Mean-Geometric Mean inequality to prove the following:

(a) $\sum_{i=1}^{n} \dfrac{i+1}{i} \geq n\sqrt[n]{n+1}$.

(b) $\sum_{i=2}^{n} \dfrac{i-1}{i} \geq n\sqrt[n]{\dfrac{1}{n}} - 1$.

Hence deduce that $n\left(\sqrt[n]{n+1} - 1\right) \leq \sum_{i=1}^{n} \dfrac{1}{i} \leq 1 + n\left(1 - \dfrac{1}{\sqrt[n]{n}}\right)$.

37. (B/C)

(a) By considering the global maximum of the function $y = \ln x - x + 1$ prove that $\ln x \leq x - 1$ whenever $x > 0$.

(b) Let x_i and t_i, $1 \leq i \leq n$, be positive numbers such that

$$x_1t_1 + x_2t_2 + \ldots + x_nt_n = t_1 + t_2 + \ldots + t_n = 1$$

Prove that

$$x_1^{t_1}x_2^{t_2}\ldots x_n^{t_n} \leq 1$$

(c) Deduce that if a_i and $p_i, 1 \leq i \leq n$, are positive numbers then

$$a_1^{p_1}a_2^{p_2}\ldots a_n^{p_n} \leq \left(\frac{a_1p_1 + a_2p_2 + \ldots + a_np_n}{p_1 + p_2 + \ldots + p_n}\right)^{p_1+p_2+\ldots+p_n}$$

38. (C/D)

(a) Write down the Maclaurin series for $\ln 2$.

(b) Prove that $\sum_{n=1}^{\infty} \dfrac{(-1)^{n+1}}{3n(3n+1)} = \dfrac{1}{3}\int_0^1 \ln(1+x^3)\,dx$. You can assume that $\int_a^b \sum_{n=1}^{\infty} a_nx^n\,dx = \sum_{n=1}^{\infty}\int_a^b a_nx^n\,dx$, for a convergent series $\sum_{n=1}^{\infty} a_nx^n$.

(c) Let

$$S_1 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \ldots$$

$$S_2 = 1 + \frac{1}{2} + \frac{1}{3} - \frac{1}{4} - \frac{1}{5} - \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} - \frac{1}{10} - \frac{1}{11} - \frac{1}{12} + \ldots$$

Prove that $S_1 + S_2 = 2 + \dfrac{2}{3}\int_0^1 \ln(1+x^3)\,dx$.

Hence deduce that $S_2 = \dfrac{1}{3}\ln 2 + \dfrac{2\pi}{3\sqrt{3}}$

39. (B/C)

(a) Given that z is a complex number with its conjugate denoted by z^*. Solve the equation $z^*z + z + z^* + i(z + z^*) = 7 + 2i$. Write down the modulus $|z|$ of the solution(s).

(b) The equation in part (a) is changed to $|z|z+z+|z|+i(z+|z|) = 7+2i$. Show that there are no solutions to this equation with z having the same modulus as in part (a).

(c) Prove that any solution z of $|z|z + z + |z| + i(z + |z|) = 7 + 2i$ must have $1.5 < |z| < 2$.

40. (D/E) Given that $a = \cos\left(\frac{2\pi}{7}\right) + i\sin\left(\frac{2\pi}{7}\right)$.

(a) Find the quadratic equation whose roots are $(a + a^2 + a^4)$ and $(a^3 + a^5 + a^6)$.

(b) By considering the root $(a+a^2+a^4)$ of the quadratic equation in (a) show that

$$\sin\left(\frac{2\pi}{7}\right) + \sin\left(\frac{4\pi}{7}\right) + \sin\left(\frac{8\pi}{7}\right) = \frac{\sqrt{7}}{2}$$

(c) Show also that

$$\sin^3\left(\frac{2\pi}{7}\right) + \sin^3\left(\frac{4\pi}{7}\right) + \sin^3\left(\frac{8\pi}{7}\right) = \frac{\sqrt{7}}{2}$$

41. (D/E)

(a) Given that $\sin x \neq 0$. Use De Moivre's theorem to express $\frac{\sin 7x}{\sin x}$ as a polynomial in $\sin x$.

(b) By considering the equation $\frac{\sin 7x}{\sin x} = 0, \sin x \neq 0$, show that

$$\sin^2\left(\frac{2\pi}{7}\right) + \sin^2\left(\frac{4\pi}{7}\right) + \sin^2\left(\frac{8\pi}{7}\right) = \frac{14}{8}$$

(c) Deduce that

$$\tan^2\left(\frac{2\pi}{7}\right) + \tan^2\left(\frac{4\pi}{7}\right) + \tan^2\left(\frac{8\pi}{7}\right) = 21$$

42. (D/E)

(a) Given that $\sin x \neq 0$. Use De Moivre's theorem to express $\dfrac{\sin 7x}{\sin x}$ as a degree 6 polynomial in $\sin x$.

(b) Express the degree 6 polynomial in (a) as the product of two cubic polynomials, each with no $\sin x$ term.

(c) Carefully explain why

$$\sin\left(\frac{2\pi}{7}\right) + \sin\left(\frac{4\pi}{7}\right) + \sin\left(\frac{8\pi}{7}\right) = \frac{\sqrt{7}}{2}$$

43. (B/C)

(a) Find the roots of $x^4 + 1 = 0$. Hence, or otherwise, express $x^4 + 1$ as the product of two quadratic terms.

(b) Decompose $\dfrac{1}{x^4+1}$ into partial fractions and hence find

$$\int \frac{1}{x^4+1} \mathrm{d}x.$$

44. (C/D) Given set of 3 non mutually perpendicular vectors $(\underline{\boldsymbol{a}}, \underline{\boldsymbol{b}}, \underline{\boldsymbol{c}})$ not lying on a plane, with $\underline{\boldsymbol{a}}$ being a unit vector[1], we can construct a set of 3 mutually perpendicular vectors $(\underline{\boldsymbol{a}}, \underline{\boldsymbol{b}}', \underline{\boldsymbol{c}}')$ as follows:

* Let $\underline{\boldsymbol{b}}' = \underline{\boldsymbol{a}} + k\underline{\boldsymbol{b}}$, where k is a scalar such that $\underline{\boldsymbol{b}}'$ is perpendicular to $\underline{\boldsymbol{a}}$.
* Let $\underline{\boldsymbol{c}}' = \underline{\boldsymbol{a}} + l\underline{\boldsymbol{b}}' + m\underline{\boldsymbol{c}}$, where l and m are some scalars such that $\underline{\boldsymbol{c}}'$ is perpendicular to both $\underline{\boldsymbol{a}}$ and $\underline{\boldsymbol{b}}'$.

Prove that $k = -\dfrac{1}{\underline{\boldsymbol{b}}.\underline{\boldsymbol{a}}}$; $m = -\dfrac{1}{\underline{\boldsymbol{c}}.\underline{\boldsymbol{a}}}$; $l = \dfrac{\underline{\boldsymbol{c}}.\underline{\boldsymbol{b}}'}{(\underline{\boldsymbol{c}}.\underline{\boldsymbol{a}})(\underline{\boldsymbol{b}}'.\underline{\boldsymbol{b}}')}$

Use this method to construct a set of 3 mutually perpendicular vectors from the set

$$\left(\frac{1}{\sqrt{6}}(\underline{\boldsymbol{i}} + 2\underline{\boldsymbol{j}} + \underline{\boldsymbol{k}}), \frac{1}{\sqrt{6}}(\underline{\boldsymbol{i}} + \underline{\boldsymbol{j}} + 2\underline{\boldsymbol{k}}), \frac{1}{\sqrt{3}}(\underline{\boldsymbol{i}} + \underline{\boldsymbol{j}} + \underline{\boldsymbol{k}})\right)$$

[1] This is is simplify the calculations, the requirement is not strictly necessary.

45. (C/D) Let $(\underline{\boldsymbol{p}}, \underline{\boldsymbol{q}}, \underline{\boldsymbol{r}})$ be a set of 3 mutually perpendicular unit vectors distinct from $(\underline{\boldsymbol{i}}, \underline{\boldsymbol{j}}, \underline{\boldsymbol{k}})$ such that

$$\underline{\boldsymbol{i}} = a_1\underline{\boldsymbol{p}} + a_2\underline{\boldsymbol{q}} + a_3\underline{\boldsymbol{r}}; \quad \underline{\boldsymbol{j}} = b_1\underline{\boldsymbol{p}} + b_2\underline{\boldsymbol{q}} + b_3\underline{\boldsymbol{r}}; \quad \underline{\boldsymbol{k}} = c_1\underline{\boldsymbol{p}} + c_2\underline{\boldsymbol{q}} + c_3\underline{\boldsymbol{r}}$$

where $a_r, b_r, c_r,\ 1 \le r \le 3$, are scalars.

Prove that

$$\underline{\boldsymbol{p}} = a_1\underline{\boldsymbol{i}} + b_1\underline{\boldsymbol{j}} + c_1\underline{\boldsymbol{k}}; \quad \underline{\boldsymbol{q}} = a_2\underline{\boldsymbol{i}} + b_2\underline{\boldsymbol{j}} + c_2\underline{\boldsymbol{k}}; \quad \underline{\boldsymbol{r}} = a_3\underline{\boldsymbol{i}} + b_3\underline{\boldsymbol{j}} + c_3\underline{\boldsymbol{k}}$$

Hence deduce that if $a_r, b_r, c_r,\ 1 \le r \le 3$, are scalars such that:

$$\sum_{r=1}^{3} a_r^2 = \sum_{r=1}^{3} b_r^2 = \sum_{r=1}^{3} c_r^2 = 1 \text{ and } \sum_{r=1}^{3} a_r b_r = \sum_{r=1}^{3} b_r c_r = \sum_{r=1}^{3} c_r a_r = 0$$

then

$a_r^2 + b_r^2 + c_r^2 = 1$ when $1 \le r \le 3$ and $a_m a_n + b_m b_n + c_m c_n = 0$ in the cases $(m, n) = (1, 2), (2, 3)$ and $(3, 1)$.

46. (D/E)

(a) Express $\tan 7x$ in terms of $t = \tan x$. Hence find the polynomial equation in t satisfied by $\tan 7x = 1$.

(b) Deduce that

$$\tan\left(\frac{\pi}{28}\right) - \tan\left(\frac{3\pi}{28}\right) + \tan\left(\frac{5\pi}{28}\right) + \tan\left(\frac{9\pi}{28}\right) - \tan\left(\frac{11\pi}{28}\right) + \tan\left(\frac{13\pi}{28}\right) = 8$$

(c) Prove that $\tan\left(\frac{\pi}{2} - \theta\right) + \tan\theta = 2\sec\left(\frac{\pi}{2} - 2\theta\right)$. Hence deduce that

$$\sec\left(\frac{\pi}{7}\right) - \sec\left(\frac{2\pi}{7}\right) + \sec\left(\frac{3\pi}{7}\right) = 4.$$

47. (C/D)

(a) A polynomial equation $x^n + a_1x^{n-1} + a_2x^{n-2} + ... + a_n = 0$ has real coefficients and n distinct roots $k_i, 1 \le i \le n$.

The sum of all p^{th} powers of the roots is $S_p = \sum_{r=1}^{n} k_i^p$.

The sum of all products of $q, 1 \le q \le n$, distinct roots is Σ_q. So, for example,
$\Sigma_2 = (k_1k_2 + k_1k_3 + ... + k_1k_n) + (k_2k_3 + ... + k_2k_n) + ... + (k_{n-1}k_n)$.

Explain why
$S_n - S_{n-1}\Sigma_1 + S_{n-2}\Sigma_2 + ... + (-1)^r S_{n-r}\Sigma_r + ... + (-1)^n n\Sigma_n = 0$.

(b) Given that $n \ge 5$ and a and b are real numbers such that the equation $x^n - ax^{n-1} + b = 0$ has n distinct roots $k_i, 1 \le i \le n$.

Also given the generalised equation satisified by the S_k and Σ_m:

$S_p - S_{p-1}\Sigma_1 + S_{p-2}\Sigma_2 + ... + (-1)^r S_{p-r}\Sigma_r + ... + (-1)^p p\Sigma_p = 0....(*)$

whenever $1 \le p \le n$.

i. Find the values of Σ_q, $1 \le q \le n$.
ii. Find S_p for $1 \le p \le n-1$.
iii. Find S_p for $n \le p \le 2n-1$

48. (C/D) Given that $1 \le x \le 1 + 2A$, where A is a non-negative constant.

(a) Prove that $\dfrac{1}{1+A} - \dfrac{x-A-1}{(1+A)^2} \le \dfrac{1}{x} \le 1 - \dfrac{x-1}{1+2A}$.

(b) Deduce that $\dfrac{2A}{1+A} \le \ln(1+2A) \le \dfrac{2A(1+A)}{1+2A}$.

49. (B/C)

(a) Let $\mathrm{f}(x)$ be a positive and non-increasing function and integrable in the interval $[a, a+n]$, where $n > 2$. Show that

$$\mathrm{f}(a+1) + \mathrm{f}(a+2) + ... + \mathrm{f}(a+n) \le \int_a^{a+n} \mathrm{f}(x)\,\mathrm{d}x \le \mathrm{f}(a) + \mathrm{f}(a+1) + ... + \mathrm{f}(a+n-1)$$

(b) Deduce that

$$\sum_{r=3}^{n+2} \frac{1}{r\ln r} \le \ln\ln(n+3) - \ln\ln 3 \le \sum_{r=2}^{n+1} \frac{1}{r\ln r}$$

50. (B/C)

(a) Given that $x + a > 0$, $x + b > 0$ and $b > a$.

Show that the derivative of $\arcsin\left(\frac{x+a}{x+b}\right)$ is $\frac{\sqrt{b-a}}{(x+b)\sqrt{(2x+b+a)}}$.

Find also the derivative of $\operatorname{arccosh}\left(\frac{x+b}{x+a}\right)$.

(b) Given that $x > -1$ find

$$\int \frac{1}{(x+3)\sqrt{x+1}}\,\mathrm{d}x; \qquad \int \frac{1}{(x+1)\sqrt{x+3}}\,\mathrm{d}x$$

51. (D/E)

(a) Given that $A > 0$ and

$$I_n = \int_0^\infty x^n \mathrm{e}^{-Ax}\,\mathrm{d}x.$$

Prove that $I_n = \frac{n}{A} I_{n-1}$ and hence deduce that $I_n = \frac{n!}{A^{n+1}}$.

(b) Prove that $x^n \frac{\sinh ax}{\cosh bx} = \sum_{k=0}^{\infty} (-1)^k \left(x^n \mathrm{e}^{-((2k+1)b-a)x} - x^n \mathrm{e}^{-((2k+1)b+a)x}\right)$

(c) Hence, given that $b > |a|$, prove that

$$\int_0^\infty x^n \frac{\sinh ax}{\cosh bx}\,\mathrm{d}x = \frac{\mathrm{d}^n}{\mathrm{d}a^n} \sum_{k=0}^{\infty} \left(\frac{(-1)^k}{(2k+1)b-a} - \frac{(-1)^{k+n}}{(2k+1)b+a}\right)$$

52. (D/E)

(a) Given that $-1 < x < 1$ and that a is not a positive integer, determine the binomial expansion of $(1+x)^a$.

(b) Given that n is an even positive integer and that

$$(1+x)^n = 1 + c_1 x + c_2 x^2 + \ldots\ldots + c_{n-1}x^{n-1} + x^n$$

Prove that

$$\frac{1}{2}c_1 - \frac{1\times 3}{2\times 4}c_2 + \frac{1\times 3\times 5}{2\times 4\times 6}c_3 + \ldots\ldots. + \frac{1\times 3\ldots.\times(2n-3)}{2\times 4\ldots..\times(2n-2)}c_{n-1} = 1$$

53. (C/D)

(a) Prove that $\dfrac{1}{a+b\cos x+c\sin x}=\dfrac{\sec^2\frac{x}{2}}{(a+b)+(a-b)\tan^2\frac{x}{2}+2c\tan\frac{x}{2}}$.

(b) Deduce that

$$\int_0^{\frac{\pi}{2}}\frac{2}{(1+2\cos x+\sin x)(3+5\cos x)}\,\mathrm{d}x=\int_0^1\frac{2+2t^2}{(t-3)(t+1)(t+2)(t-2)}\,\mathrm{d}t$$

(c) Hence prove that

$$\int_0^{\frac{\pi}{2}}\frac{2}{(1+2\cos x+\sin x)(3+5\cos x)}\,\mathrm{d}x=\ln\frac{2^{\frac{8}{3}}}{3^{\frac{3}{2}}}$$

54. (A/B) a_1,a_2,a_3,a_4 are the roots of the equation $x^4-px^3+qx^2-px+r=0$. Given that $1\le i,j,k,l\le 4$ and i is distinct from j,k,l. Show that

$$\frac{(a_i+a_j)(a_i+a_k)(a_i+a_l)}{1+a_i^2}$$

has the same value for all 4 values of i.

55. (C/D) Given that n is an odd positive integer.

(a) Using mathematical induction prove that

$$I_n=\int_0^{\pi}\frac{\cos nx}{\cos x}\,\mathrm{d}x=(-1)^{\frac{n-1}{2}}\pi$$

(b)

$$J_n=\int_0^{\pi}\sin 2(n-1)x\tan x\,\mathrm{d}x=-\pi;\quad n\ge 3$$

Prove that $J_n=J_{n-2}=-\pi$

(c) Deduce that

$$K_n=\int_0^{\pi}\frac{\cos^2 nx}{\cos^2 x}\,\mathrm{d}x=K_{n-2}+2\pi=n\pi$$

56. (A/B) The roots of the equation $(1+x^2)^2=ax(1-x^2)+b(1-x^4)$, where $b\neq -1$, are also the roots of $x^3+px^2+qx+r=0$. Prove that $p^2-q^2-r^2+1=0$.

57. (A/B) The roots of the equation $x^3+px+q=0$ are a,b and c.

(a) If $b=kc$, where k is some non-zero scalar, prove that $p^3(k^2+k)^2+q^2(k^2+k+1)^3=0$.

(b) If $q\neq 0$ show that one root is $-\dfrac{q(k^2+k+1)}{p(k^2+k)}$ and find the other two roots in terms of p,q and k.

58. (C/D)

(a) Prove that if $0 < a < \pi$ then

$$\int_0^{\pi} \frac{1}{1+\cos a \cos x} \mathrm{d}x = \frac{a}{\sin a}$$

(b) Find the value of

$$\int_0^{\pi} \frac{1}{1+\cos a \cos x} \mathrm{d}x$$

when $\pi < a < 2\pi$.

59. (A/B) Prove that $\prod_{r=1}^{n} \left(\cos\left(\frac{2\pi}{n}\right) + \sin\left(\frac{2\pi}{n}\right) \cot\left(\frac{(2r-1)\pi}{n}\right)\right) = 1$

60. (B/C) Find the integer n for which

$$\prod_{r=1}^{45} (1+\tan r^\circ) = 2^n$$

61. (C/D) Find x and y as mutliples of π given that

(a) $\sum_{r=0}^{10} \sin\left(\frac{(4r+2)\pi}{23}\right) = \frac{1}{2}\tan x.$

(b) $\sum_{r=0}^{10} (-1)^r \sin\left(\frac{(4r+2)\pi}{23}\right) = -\frac{1}{2}\tan y.$

62. (C/D) Given that the real polynomial $\mathrm{f}(x) = 3x^5 + ax^4 + bx^3 + cx^2 + dx + e$ satisfies the following:

$\mathrm{f}(x)$ has remainder $4x+5$ when divided by $(x^2-1)(x-1)$.
$\mathrm{f}(x)$ has the same remainder when divided by x^2+1 or by x^2+3x+3.

Find the values of a, b, c, d and e.

63. (B/C)

(a) If a and b are positive numbers then show that $\frac{a+b}{4} - \frac{ab}{a+b} \geq 0.$

(b) If a, b and c are positive numbers then show that

$$\frac{a+b+c}{2} - \frac{ab}{a+b} - \frac{bc}{b+c} - \frac{ca}{c+a} \geq 0.$$

64. (D/E) The real valued function $\mathrm{f}(m,n)$ defined on pairs of non-negative integers satisfies the following:
$\mathrm{f}(m,n) = \mathrm{f}(m-1,n) + \mathrm{f}(m,n-1)$, when both m and n are positive.
$\mathrm{f}(m,0) = 1$, when $m \geq 0$ and $\mathrm{f}(0,n) = 0$, when $n \geq 1$.

(a) Show that $\mathrm{f}(m,1) = m$ and $\mathrm{f}(m,2) = \dfrac{m(m+1)}{2}$.

(b) Find $\mathrm{f}(m,3)$ and hence make a conjecture about the value of $\mathrm{f}(m,k)$ whenever $k \geq 1$.

(c) If $-1 \leq x+y \leq 1$ prove that $\sum\limits_{m=0}^{\infty}\sum\limits_{n=0}^{\infty} \mathrm{f}(m,n)x^m y^n = \dfrac{1-y}{1-x-y}$.

(d) Hence prove the conjecture in part (b).

65. (C/D) Prove that

$$2\sin\left(1+\frac{1}{2}+\frac{1}{4}+\ldots\ldots+\frac{1}{2^{n-1}}\right)\frac{\pi}{4} = \sqrt{2+\sqrt{2+\ldots\ldots+\sqrt{2}}},$$

where there are n 2's in the right hand side.

Let $N_n = \sqrt{2-\sqrt{2+\ldots\ldots+\sqrt{2}}}$, where there are n 2's under the square root sign. Prove that $\dfrac{N_n}{N_{n-1}} > \dfrac{1}{2}$.

66. (C/D) By writing $z = \sqrt{x} + i\sqrt{y}$, solve the system of equations:

$$\sqrt{3x}\left(1+\frac{1}{x+y}\right) = 2$$
$$\sqrt{7y}\left(1-\frac{1}{x+y}\right) = 4\sqrt{2}$$

67. (C/D) Given that θ is a real number and that $f(x) = \dfrac{2\sin\theta}{x^2 - 2x\cos\theta + 1}$.

(a) Express the denominator in $f(x)$ as a product of two linear factors and then decompose $f(x)$ into two partial fractions. Deduce that the Maclaurin series of $f(x)$ is $\sum_{k=0}^{\infty} 2\sin(n+1)\theta x^n$.

(b) Deduce that $\dfrac{1}{\sin\theta} = \sum_{k=0}^{\infty}\cos^n\theta\sin(n+1)\theta$ ....(*)

(c) When $\theta = \frac{\pi}{3}$ prove that $\sum_{k=0}^{\infty}\cos^n\theta\sin(n+1)\theta$ is a sum of two geometric series. Hence justify (*) in the case $\theta = \frac{\pi}{3}$.

68. (C/D) The sequence of distinct real numbers $x_i, i \geq 0$, satisfies the recurrence relation

$$x_{i+1}^2 - 7x_{i+1}x_i + x_i^2 + 9 = 0, \text{ where } x_0 = 1 \text{ and } x_1 = 5$$

Prove the following statements

(a) Each x_i is a positive integer.

(b) $x_i x_{i-1} - 1$ is the square of an integer for $i \geq 0$.

(c) $\sqrt{5x_i^2 - 4}$ is an integer for $i \geq 0$.

69. (C/D) The sequence of real numbers $x_i, i \geq 1$, satisfies the recurrence relation

$$x_{i-1}^2 - x_i x_{i-2} + 2 = 0, \ i \geq 3, \text{ witn } x_1 = x_2 = 1$$

Find an expresion for $\dfrac{x_i}{x_{i-1}}$. Hence prove that each x_i is a positive integer.

70. (A/B) A sequence of sums of squares is given below:

$$3^2 + 4^2 = 5^3; \ \ 5^2 + 12^2 = 13^2; \ \ 7^2 + 24^2 = 25^2; \ \ 9^2 + 40^2 = 41^2; \ldots\ldots$$

Write down the next two terms of the sequence.

Express algebraically the n^{th} term of the sequence and verify its validity.

71. (B/C)

(a) Confirm that $\sin\left(\dfrac{\pi}{2k}\right)\sin\left(\dfrac{3\pi}{2k}\right) \ldots\ldots \sin\left(\dfrac{k\pi}{2k}\right) = \dfrac{1}{2^{(k-1)/2}} \ldots\ldots(\alpha)$

for $k = 1$ and 3.

And that $\sin\left(\dfrac{\pi}{2k}\right)\sin\left(\dfrac{3\pi}{2k}\right) \ldots\ldots \sin\left(\dfrac{(k-1)\pi}{2k}\right) = \dfrac{1}{2^{(k-1)/2}} \ldots\ldots(\beta)$

for $k = 2$ and 4.

(b) Let $x = 2\sin\left(\dfrac{\pi}{10}\right)\sin\left(\dfrac{3\pi}{10}\right)$. Prove that $2x^2 + 3x - 2 = 0$.

Deduce that the formula is (α) is true also for $k = 5$.

72. (C/D)

(a) Let $E = \sin x \sin 3x \sin 5x \sin 7x$, where $8x = \dfrac{\pi}{2}$.
Show that $8E = \dfrac{1}{\sqrt{2}}$.

(b) Let $k = 2^n$, where n is some positive integer. Show that

$$\sin\left(\frac{\pi}{2k}\right)\sin\left(\frac{3\pi}{2k}\right)\ldots\ldots\sin\left(\frac{(k-1)\pi}{2k}\right) = \frac{1}{2^{(k-1)/2}}$$

73. (B/C) Given that a, b and c are distinct real numbers such that each is less than the sum of the other two. Prove that each is greater than the distance between the other two.

Prove also that

$$a + b - c < \frac{2ab}{c}$$

74. (C/D)

(a) If $\mathrm{f}(x) = \sinh x - x$ show that $\mathrm{f}(0) = 0$ and that $\mathrm{f}'(x) \geq 0$ when $x \geq 0$. Deduce that $\sinh x > x$ when $x > 0$.

Similarly prove that that $x > \sinh^{-1} x$ when $x > 0$.

(b) Prove that $\mathrm{g}(x) = x^{-1}\sinh x$ is an increasing function in the domain $x > 0$.

(c) Deduce that $\sinh x \sinh^{-1} x > x^2$ when $x > 0$.

75. (C/D) Using the identity $\mathrm{e}^{ix} = \cos x + i \sin x$, where $i = \sqrt{-1}$, find

(a) $S_n = \cos x \cos x + \cos 2x \cos^2 x + \cos 3x \cos^3 x + \ldots\ldots + \cos nx \cos^n x$.

(b) $T_n = \sin x \cos x + \sin 2x \cos^2 x + \sin 3x \cos^3 x + \ldots\ldots + \sin nx \cos^n x$.

(c) $S_\infty = \cos x \cos x + \cos 2x \cos^2 x + \cos 3x \cos^3 x + \ldots\ldots\ldots\ldots$

(d) $T_\infty = \sin x \cos x + \sin 2x \cos^2 x + \sin 3x \cos^3 x + \ldots\ldots\ldots\ldots\ldots$

For parts (c) and (d) you can assume that $\mathrm{e}^{inx} \cos^n x \to 0$ as $n \to \infty$.

Deduce that
$U_\infty = \sin x \sin x - \cos 2x \sin^2 x - \sin 3x \sin^3 x + \cos 4x \sin^4 x + \sin 5x \sin^5 x$
$- \cos 6x \sin^6 x - \sin 7x \sin^7 x + \ldots\ldots\ldots\ldots\ldots\ldots\ldots = 0$.

76. (B/C)

(a) Find $S_n = \dfrac{1}{2} + \dfrac{1}{1 \times 3} + \dfrac{1}{1 \times 2 \times 4} + \ldots\ldots + \dfrac{1}{1 \times 2 \times 3 \times \ldots. \times (n-1) \times (n+1)}$.

(b) If $0 < x < 1$ find the sum of the infinite series

$$T_\infty = S_1 x + S_2 x^2 + S_3 x^3 + \ldots.. + S_n x^n + \ldots\ldots$$

77. (C/D) Suppose that the cubic equation with real coefficients $ax^3 + bx^2 + cx + d = 0$ can be written in the form

$$L(x+s)^3 + M(x+t)^3 = 0$$

with L and M non-zero.

(a) Write a, b, c and d in terms of the numbers L, M, s and t.

(b) Verify that s and t are the roots of the quadratic equation

$$(ac - b^2)y^2 - (ad - bc)y + (bd - c^2) = 0$$

(c) Deduce a condition that implies that the roots of the cubic equation are not all real.

78. (B/C) Evaluate the integral

$$I = \int_0^1 \frac{1 + x^{\frac{1}{2}}}{1 + x^{\frac{1}{3}}}\,\mathrm{d}x.$$

79. (A/B) The cubic equation with real coefficients $x^3 + ax^2 + bx + a = 0$, with $b \neq 1$, has three real roots p, q and r.

Prove that $\tan^{-1} p + \tan^{-1} q + \tan^{-1} r = n\pi$, where n is some integer.

80. (B/C) Find the sum of the infinite series

$$1 + \frac{2^2}{1!} + \frac{3^2}{2!} + \ldots.. + \frac{r^2}{(r-1)!} + \ldots...$$

81. (B/C) A triangle has sides of length a, b and c. Prove that

$$3(ab + bc + ca) \leq (a + b + c)^2 \leq 4(ab + bc + ca).$$

82. (A/B). The angle biscector of angle C in triangle ABC meets AB in D. Prove that $CD = \dfrac{2ab\cos C/2}{a+b}$.

83. (A/B) It is given that p is a non-zero positive integer and that the sequence $a_i; i \geq 0$ satisfies $a_0 = 0$ and $a_{i+1} = pa_i + \sqrt{(p^2-1)a_i^2 + 1}$ for $i \geq 0$.

(a) Prove that the relation $a_{i+1} = pa_i + \sqrt{(p^2-1)a_i^2 + 1}$ is equivalent to $a_{i+1}^2 - 2pa_ia_{i+1} + a_i^2 - 1 = 0$.

(b) Use the relation $a_{i+1}^2 - 2pa_ia_{i+1} + a_i^2 - 1 = 0$ to find a relation of the type $Ka_{i+1} + 2pLa_i + Ma_{i-1} = 0$, where K, L and M are integers.

(c) Hence prove that each a_i is an integer and that $2p$ divides every $a_{2j}, j \geq 1$.

84. (B/C) Given that $\sum\limits_{n=0}^{\infty} \dfrac{1}{n!} = \mathrm{e}$.

(a) By considering n^3 and n^4 as $n^3 = n^3 - n^2 + n^2 - n + n$ and $n^4 = n^4 - n^3 + n^3 - n^2 + n^2 - n + n$ show that

$$\sum_{n=0}^{\infty} \frac{n^3}{n!} = \left(\sum_{n=3}^{\infty} \frac{1}{(n-3)!} + \sum_{n=2}^{\infty} \frac{2}{(n-2)!} + \sum_{n=2}^{\infty} \frac{1}{(n-2)!} + \sum_{n=1}^{\infty} \frac{1}{(n-1)!} \right)$$

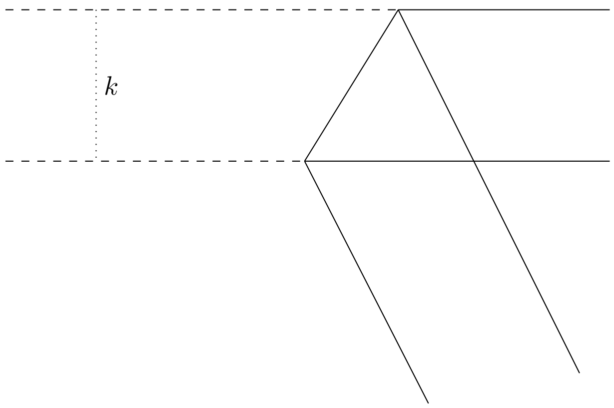
and

$$\sum_{n=0}^{\infty} \frac{n^4}{n!} = \left(\sum_{n=4}^{\infty} \frac{1}{(n-4)!} + \sum_{n=3}^{\infty} \frac{3}{(n-3)!} + \sum_{n=3}^{\infty} \frac{2}{(n-3)!} + \sum_{n=2}^{\infty} \frac{4}{(n-2)!} \right) + \left(\sum_{n=3}^{\infty} \frac{1}{(n-3)!} + \sum_{n=2}^{\infty} \frac{2}{(n-2)!} + \sum_{n=2}^{\infty} \frac{1}{(n-2)!} + \sum_{n=1}^{\infty} \frac{1}{(n-1)!} \right).$$

Hence determine $\sum\limits_{n=0}^{\infty} \dfrac{n^3}{n!}$ and $\sum\limits_{n=0}^{\infty} \dfrac{n^4}{n!}$ in terms of e.

(b) Given that k is a positive integer. Use the results of (a) to make a conjecture about the sum of $\sum\limits_{n=0}^{\infty} \dfrac{n^k}{n!}$ in terms of k and e. Substitute $k = 3$ and 4 to test the conjecture.

85. (A/B) A infinitely long rectangular strip of paper k cm wide is folded so that it does not completely overlap (as shown in the section diagram below). Prove that the minimum area of overlap is $\dfrac{k^2}{2}$ sq.cm.



86. (B/C)

(a) Show that the family of parabolae with equations $y = x^2 + 2ax + a$, where a is any real number, have a common point and that their minimum points lie on a parabola.

(b) For the parabolae with equations $y = x^2 + 2ax + a$ that have three intercepts on the axes, determine the equation of the circle that passes through these three intercepts. Hence identify the coordinates of any common point(s) of the family of these circles.

87. (A/B) In a triangle ABC with the usual conventions of sides of lengths a, b, c and angles A, B, C.
Given that $A = 3B$. Show that $(a^2 - b^2)(a - b) = bc^2$.

88. (C/D) An arithmetic series of 100 terms has the property that the sum of the even numbered terms (i.e. the 2^{nd}, 4^{th}, ..., 98^{th} and 100^{th} terms) is 1, while the sum of odd numbered terms is -2. Find the sum of the squares of all 100 terms.

89. (C/D) The family of parabolae with equations $y = x^2 + px + q$ have three intercepts on the axes. Show that the equation of the circle that passes through these three intercepts is $x^2 + y^2 + px - (1 + q)y + q = 0$. Hence identify the coordinates of any common point(s) of the family of these circles.

90. (C/D) Show that $\left(x - \frac{1}{x}\right)^{\frac{1}{2}} + \left(1 - \frac{1}{x}\right)^{\frac{1}{2}} = \dfrac{x - 1}{\left(x - \frac{1}{x}\right)^{\frac{1}{2}} - \left(1 - \frac{1}{x}\right)^{\frac{1}{2}}}$.

Hence solve the equation $x = \left(x - \frac{1}{x}\right)^{\frac{1}{2}} + \left(1 - \frac{1}{x}\right)^{\frac{1}{2}}$.

91. (B/C) Given that $f(x) = x^2 - 2ax - a^2 - \frac{3}{4}$, where a is a real constant. Find the range of values of a for which $|f(x)| < 1$ in the interval $[0, 1]$.

92. (B/C)

(a) Find the positive integer n such that $n(n + 1) = 40743420$.

(b) $S = 1 + \dfrac{1}{1 + \frac{1}{3}} + \dfrac{1}{1 + \frac{1}{3} + \frac{1}{6}} + \dfrac{1}{1 + \frac{1}{3} + \frac{1}{6} + \frac{1}{10}} + \ldots\ldots + \dfrac{1}{1 + \frac{1}{3} + \frac{1}{6} + \ldots.. + \frac{1}{40743420}}$

Prove that $S > 1009.5$.

93. (B/C) A and B are the roots of the equation $x^2 - x - 1 = 0$ and

$$a_n = \frac{A^n - B^n}{A - B}, \text{ for } n \geq 0.$$

Prove that $a_{n+2} = a_{n+1} + a_n$ for $n \geq 2$.

94. (C/D) Show that $E = a^5(c - b) + b^5(a - c) + c^5(b - a)$ can be written as $F\left((a + b + c)(a^2 + b^2 + c^2) + abc\right)$ where the factor F has to be determined.

95. (B/C) Find the six solutions of the equation $(1+x^2)(1+x^4)=4x^3$.

96. (B/C) Find the real solutions of the equation $x^2+2x\sin(xy)+1=0$.

97. (C/D) Given that each of a, b and c are non-negative numbers.

    Prove that $\frac{a}{b+c}+\frac{b}{a+c}+\frac{c}{a+b}\geq 0$.

98. (C/D) Use the generalised remainder theorem to find the value of k such that $x^5+y^5+z^5+k(x^3+y^3+z^3)(x^2+y^2+z^2)$ has factor $(x+y+z)$.

    For this value of k show that $(x+y+z)^2$ is also a factor of the expression.

99. (C/D) Given that $\sin x+\sin y=a$ and $\cos x+\cos y=b$.

    (a) Prove that $\tan\frac{x+y}{2}=\frac{a}{b}$ and that $2+2\cos(x-y)=a^2+b^2$.

    (b) Deduce that

    $$A\tan^2\frac{x}{2}+A\tan^2\frac{y}{2}+A\tan^2\frac{x}{2}\tan^2\frac{y}{2}+\tan\frac{x}{2}\tan\frac{y}{2}-A=0,$$

    where $A=4-a^2-b^2$.

    (c) Also prove that $\tan\frac{x-y}{2}=\sqrt{\frac{4-a^2-b^2}{a^2+b^2}}$.

    (d) Find the values of $\tan\frac{x}{2}$ and $\tan\frac{y}{2}$ in the case that $a=b=\sqrt{2}$.

100. (B/C) A sequence is defined by $u_1=1,\ u_2=N$ and $u_n=\frac{u_{n-1}^2+2}{u_{n-2}}$, for $n\geq 3$.

    Show that $\frac{u_{n-1}}{u_{n-2}}=\frac{u_n+u_{n-2}}{u_{n-1}+u_{n-3}}$ for $n\geq 3$.

    Hence show that $u_n=au_{n-1}+bu_{n-2}$ for $n\geq 3$, where the values a and b have to be determined.

    For what values of N is each u_n an integer?

101. (B/C) An increasing sequence is defined by $u_1=u_2=u_3=1$ and

    $$u_{n+1}=\frac{1+u_nu_{n-1}}{u_{n-2}},$$

    for $n\geq 4$.

    Prove that $u_{n+1}=au_{n-1}+bu_{n-3}$ for $n\geq 3$, where a and b are integers to be found. Deduce that each u_n an integer.

102. (C/D)

(a) Prove that $\sin 5x = 16\sin^5 x - 20\sin^3 x + 5\sin x$.

(b) Express $\cos 36° - \cos 72°$ in terms of $\sin 36°$. Hence find the exact value of $\cos 36° - \cos 72°$.

103. (B/C) Given that $n > 2$ and $r > 0$ are integers. Prove that $\left(\dfrac{n+\sqrt{n^2-4}}{2}\right)^r$ can be written as $\dfrac{k_r+\sqrt{k_r^2-4}}{2}$, where k_r is an integer, for $r = 2, 3, 4, 5, 6$.

104. From the previous problem we get a strong conjecture that $\left(\dfrac{n+\sqrt{n^2-4}}{2}\right)^r$ can be written as $\dfrac{k_r+\sqrt{k_r^2-4}}{2}$, where k_r is an integer, for $r \geq 1$. To prove the conjectue without using brute force algebra is the essence of this problem.

(a) Let r be a positive integer. In the case the r is even show that $A^r + B^r = (A+B)^r - {}^rC_1 AB(A^{r-2}+B^{r-2}) - {}^rC_2(AB)^2(A^{r-4}+B^{r-4}) - .... - {}^rC_{r/2}(AB)^{r/2}$.

In the case the r is odd show that $A^r + B^r = (A+B)^r - {}^rC_1 AB(A^{r-2}+B^{r-2}) - {}^rC_2(AB)^2(A^{r-4}+B^{r-4}) - .... - {}^rC_{(r-1)/2}(AB)^{(r-1)/2}(A+B)$.

(b) Prove that if A and B are the roots of the equation $x^2 - nx + 1 = 0$ then $A^r + B^r$, where r is a positive integer, is always an integer. Hence prove the conjecture made above.

105. (A/B) Find all solutions of the equation $\dfrac{x^3}{\sqrt{4-x^2}} + x^2 - 4 = 0$.

106. (B/C)

(a) Given that $x \neq 0$. Evaluate the sum $f(x) = x^{-1} + x^{-2} + ... + x^{-n}$ and then find $\dfrac{\mathrm{d}f(x)}{\mathrm{d}x}$.

(b) Hence prove that $\dfrac{1}{2} + \dfrac{3}{2^2} + \dfrac{5}{2^3} + \dfrac{7}{2^3} + ... + \dfrac{2n-1}{2^n} = \dfrac{a(2^n-1)-bn}{2^n}$,

where a and b are integers to be found.

107. (B/C) Let u_r be a sequence defined by $u_1 = 1$ and $u_r = 1 + u_1 u_2 .... u_{r-1}$, whenever $r > 1$.

Prove that $\sum_{r=1}^{n} \frac{1}{u_r} = m - \frac{1}{u_1 u_2 ..... u_n}$, where m is an integer to be found.
Deduce the value of $\sum_{r=1}^{\infty} \frac{1}{u_r}$.