University of Leicester, School of Mathematics and Actuarial Science/AMSP y12 challenge problems.

Degree of difficulty A (easier) to E (hardest). Sources of problems include: the history of mathematics, junior national mathematics olympiads, past STEP papers, published online problems, etc.

1. (A) Show that the sum of all integers formed by using the digits a, b, c and d is divisible by the integer $(a+b+c+d)$.

2. (B/C) It is given that $\dfrac{2x-3y}{x+2y} = 3$. Determine the value of $\dfrac{2x+y}{3x+10y}$.

3. (A) Given that $x^2 + xy = 20$ and $y^2 + xy = 30$. Find x and y.

4. (B) It is given that $x^2 + 6x + y^2 - 4y + 13 = 0$. Determine the value of $x + y$.

5. (C) Given that m is a real number. Find the values of m for which:
   (a) $x^2 - 4x - 2m(m^2+1) = 0$ has equal roots.
   (b) $\left(x^2 - 4x - 2m(m^2+1)\right)\left(x^2 - 2mx - 4(m^2+1)\right) = 0$ has exactly two different roots.

6. (D/E) Positive integers l, m and n are such that the roots of the 3 equations

$$x^2 - 2lx + m = 0 \qquad (1)$$
$$x^2 - 2mx + n = 0 \qquad (2)$$
$$x^2 - 2nx + l = 0 \qquad (3)$$

are also all positive integers. Find all such sets of positive integers (l, m, n).

7. (C/D) Let $f(x) = x^2 + bx + c + 1$ and $g(x) = x^3 + bx^2 + cx + 1$.
   (a) Find the condition satisfied by the constants b and c so that the equations $f(x) = 0$ and $g(x) = 0$ have a common root.
   (b) Given that b and c satisfy this condition find the sum of the distinct roots of the equation $f(x)g(x) = 0$.

8. (E) a_n is given by the formula $a_n = 1 + \dfrac{6n^2 + 12n + 8}{n^3 + 1}$, and b_n by the formula $b_n = a_0 a_1 \cdots a_{n-1} a_n$. Calculate the values of b_1, b_3 and b_{99}.

9. (C/D)
   (a) Show that the reflection of the point (p, q) in the line $y = mx$ is the point $\left(\dfrac{(1 - m^2)p + 2mq}{m^2 + 1}, \dfrac{2mp + (m^2 - 1)q}{m^2 + 1}\right)$.
   (b) The side AB of a triangle ABC passes through a fixed point $P(a, b)$. The side AC is bisected at right angles by the line $x + y = 0$ while the side BC is bisected at right angles by the line $x = 9y$. Show that C lies on the circle $4x^2 + 4y^2 + (4b - 5a)x + (4a + 5b)y = 0$.

   Hints:
   Suppose the line joining the point $N(p, q)$ to its image Q meets the mirror line $y = mx$ at M. Since M lies on $y = mx$ its coordinates are (r, mr) for some number r.

10. (C/D) A pyramid stands on horizontal ground. Its base is an equilateral triangle with sides of length a, the other three sides of the pyramid are of length b and its volume is V. Given that the formula for the volume of any pyramid is $\frac{1}{2} \times$ area of base $\times$ height, show that

$$V = \frac{1}{12} a^2 \sqrt{3b^2 - a^2}$$

    The pyramid is then placed so that a non-equilateral face lies on the ground. Show that the new height, h, of the pyramid is given by

$$h = \frac{a\sqrt{3b^2 - a^2}}{\sqrt{4b^2 - a^2}}$$

11. (B/C) It is given that $x - y = d \neq 0$ and that $x^2 - y^2 = (x - y)^3$. Express each of x and y in terms of d.

    Hence find a pair of integers m and n, where $m > n > 100$, satisfying

$$m - n = \left(\sqrt{m} - \sqrt{n}\right)^3$$

12. (C/D) In the Cartesian plane points A, B and P have, respectively, coordinates $(a, 0), (0, b)$ and (s, t), where each of a, b, s and t are positive. X, Y and N are, respectively, t3+he feet of the perpendiculars from P to the x axis, y axis and AB. Determine the coordinates of N in terms of a, b, t and s. Show that if N lies on the line XY then $\left(\frac{t-b}{s}\right)\left(\frac{t}{s-a}\right) = -1$.

13. (B/C) Two distinct points A and B lie on the line $y = 2x - 17$ and, repectively, have coordinates $(a, 2a - 17)$ and $(b, 2b - 17)$, where $b > a$. The distinct points C and D are such that $ABCD$ is a square. Find the coordinates of C and D in terms of a and b.
If, further, the points C and D lie on the curve $y = x^2$ prove that
$5a - 3b = 2$.

14. (A/B) Points on the quadratic curve $y = x^2$ have coordinates (a, a^2). Given that $a \neq \frac{1}{2}$, find the coordinates (X, Y) of the intersection poit P of the tangent T at (a, a^2) and the normal N at $(-a, a^2)$. Show that P lies on the line $y = ax + \frac{1}{2} + \frac{1}{8a^2 - 1}$.

    What can you say about the tangent T and normal N in the case $a = \frac{1}{2}$?

15. (A/B) The first four terms of a sequence I are: $a, a + d, a + 2d, a + 3d$. The first four terms of another sequence II are: $A(\neq 1), Ar, Ar^2, Ar^3$. The eight terms satisfy:

$$a + A = 27 \tag{4}$$
$$(a + d) + (Ar) = 27 \tag{5}$$
$$(a + 2d) + (Ar^2) = 39 \tag{6}$$
$$(a + 3d) + (Ar^3) = 87 \tag{7}$$

    By using the substitution $a = 27 - A$ find the all eight terms.

16. (A/B)

    (a) Expand $\left(x - \frac{1}{x}\right)^2$.

    (b) By dividing the equation $5x^4 - 19x^3 - 34x^2 + 19x + 5 = 0$ by x^2 and using a suitable substitution reduce it to a quadratic equation. Hence find all real roots of $5x^4 - 19x^3 - 34x^2 + 19x + 5 = 0$.

17. (B/C) Given that

$$5x^2 + 2y^2 - 6xy + 4x - 4y = a(x - y + 2)^2 + b(cx + y)^2 + d.$$

Find the values of the constants a, b, c and d.

Solve the simultaneous equations

$$5x^2 + 2y^2 - 6xy + 4x - 4y = 9$$

$$6x^2 + 3y^2 - 8xy + 8x - 8y = 14$$

You may assume $6x^2 + 3y^2 - 8xy + 8x - 8y$ can be written as $\alpha(x - y + 2)^2 + (cx + y)^2 + \delta$, where α and δ are some constants.

18. (A/B) Given that n is a positive integer. Use factorisation to directly prove the following

    (a) $(n^5 - n^3)$ is divisible by 24.

    (b) If $n - 1$ is a multiple of 3 then $n^3 - 1$ is divisible by 9.

19. (B/C) The multiples of 3 and 5 are removed from the list of numbers from 1 to n. Then the average m of the numbers that remain is calculated.

    Find m in the cases i) n=14 and ii) n=29. Make a conjecture from these findings and predict the value of m when $n = 2999999$.

20. (A/B) Show that the number $3^2 \times 5^3$ has exactly 10 proper factors (a proper factor of a natural number N is a factor that is not 1 or N). Determine how many other numbers of the form $3^m \times 5^n$, where m and n are positive integers, have exactly 10 proper factors.

21. (C/D) N is the smallest positive integer that has exactly 426 proper factors. It is given that N has 3 prime factors. Determine N as a product of prime factors.

22. (B/C) Given that $160^2 = 25600$. Find the integer m such that $m^2 < 32899 < (m + 1)^2$. Next find the value of small integer n such that $(m + n)^2 - 32899$ is a square number. Now express 32899 as product of two primes. Hence show that there are exactly two values of n for which $(m + n)^2 - 32899$ is a square number.

23. A 14$^{\text{th}}$ century series for π is

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots.. + \frac{(-1)^n}{2n+1} + \ldots\ldots. \quad (*)$$

Show that replacing:

* 1 by $\left(1 - \frac{1}{2}\right)$

* **And** $\frac{(-1)^n}{2n+1}$ by $(-1)^{n+1}\left(\frac{1}{4n-2} - \frac{1}{2n+1} + \frac{1}{4n+2}\right)$

leaves the series unchanged. And show that the substitution results in the series:

$$\frac{\pi}{4} = \frac{1}{2} + \frac{1}{2^2-1} - \frac{1}{4^2-1} + \ldots.. - \frac{(-1)^n}{(2n)^2-1} + \ldots\ldots.$$

24. As in Q23 we have:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots.. + \frac{(-1)^n}{2n+1} + \ldots\ldots. \quad (*)$$

Show that replacing:

* 1 by $\left(1 - \cfrac{1}{2\times 1 + 2 + \cfrac{4}{2\times 1+2}}\right)$ **and**, for $n \geq 1$,

$\frac{(-1)^n}{2n+1}$ by $(-1)^{n+1}\left(\cfrac{1}{2(2n-1)+2+\cfrac{4}{2(2n-1)+2}} - \cfrac{1}{2n+1} + \cfrac{1}{2(2n+1)+2+\cfrac{4}{2(2n+1)+2}}\right)$

leaves the series unchanged. And show that the substitution results in the series:

$$\frac{\pi}{4} = \frac{4}{4\times 1 + 1^5} - \frac{4}{4\times 3 + 3^5} + \ldots\ldots.. + \frac{4(-1)^n}{4(2n+1)+(2n+1)^5} + \ldots\ldots..$$

25. (C/D) A common tangent to two curves is a line that is tangent to the two curves at the same time.

    Find the explicit equation of the common tangent to the two curves $y = x^2 + ax + b$ and $y = x^2 + bx + a$, where $a \neq b$.
    Also show that the absolute value of difference of the x coordinates of the points of contact of this common tangent with each of the two curves is $\left|\dfrac{b-a}{2}\right|$.

    Identify the value of $a + b$ for which the absolute value of difference of the y coordinates of the points of contact of this common tangent with the two curves is zero. Show that in this case the two curves are horizontal translations of each other, and give the translation factor.

    Hints:
    Let $y = mx + c$ be the common tangent line.
    Then $mx + c = x^2 + ax + b$ and $mx + c = x^2 + bx + a$ must each have repeated roots.
    So we require $(a - m)^2 = 4(b - c)$ and $(b - m)^2 = 4(a - c)$.

26. (C/D) $\alpha > \beta$ are real roots of the quadratic equation $x^2 + 2bx - a^3 = 0$, where a and b are real numbers with $a^3 + b^2 > 0$.

    Determine α and β in terms of a and b.

    By using the identity $(p + q)^3 = p^3 + q^2 + 3pq(p + q)$ show that $\alpha^{\frac{1}{3}} + \beta^{\frac{1}{3}}$ is a root of $x^3 + 3ax + 2b = 0$.

    Hence show that $\left(\dfrac{1+\sqrt{\frac{31}{27}}}{2}\right)^{\frac{1}{3}} + \left(\dfrac{1-\sqrt{\frac{31}{27}}}{2}\right)^{\frac{1}{3}}$ is a root of the equation $x^3 + x - 1 = 0$.

27. (B/C) Use the fact that $(p + q)^3 = p^3 + q^3 + 3pq(p + q)$ and the factor theorem to identify a root, in terms of a and b, of the equation $x^3 - 3abx - (a^3 + b^3) = 0$.
    Find a root in surd form of the equation $4x^3 - 6x - 3 = 0$ by first making it equivalent to the form $x^3 - 3abx - (a^3 + b^3) = 0$ and then forming a quadratic equation in a^3.

28. (C/D) The roots of a cubic equation $f(x) = 0$ are a, b and c. The roots of the equation $f'(x) = 0$ are p and q. Use the theory of roots of polynomial equations to prove that $(a - p)(a - q) = \dfrac{1}{3}(a - b)(a - c)$. Find similar relations for $(b - p)(b - q)$ and $(c - p)(c - q)$ in terms of a, b and c.

29. (B/C)

(a) Given that the polynomial $p(x)$ is given by $p(x) = (x-1)(x-2)(x-3)q(x) + r(x)$, where $q(x)$ and $r(x)$ are polynomials with $r(x)$ a quadratic polynomial. When $p(x)$ is divided by $(x-1), (x-2)$ and $(x-3)$, repectively, the remainders are 3, 1 and 5 respectively. Find $r(x)$.

(b) Deduce that a polynomial $P(x)$ of degree $(n+1)$, where n is a given positive integer, such that, for each integer a satisfying $0 \leq a \leq n$, the remainder when $P(x)$ is divided by $(x-a)$ is a.

30. (A/B) Given that

$$x^3 - 5x^2 + 2x^2y + xy^2 - 8xy - 3y^2 + 6x + 6y$$

Use the factor theorem to identify two factors of the form $(x+m)$ and $(x+ny)$, where m and n are integers, of the expression above. Hence factorise the expression completely.

31. (C/D) The equation $x^3 + 4x - 1 = 0$ has roots p, q and r. Use the substitution $y = \dfrac{1}{1+x}$

to prove that the equation with roots $(p+1)^{-1}, (q+1)^{-1}$ and $(r+1)^{-1}$ is:

$$6y^3 - 7y^2 + 3y - 1 = 0$$

Use the theory of roots of polynomial equations and the expansion identity for $(a+b+c)^2$ to find the values of $(p+1)^{-k} + (q+1)^{-k} + (r+1)^{-k}$ for $k = 1$ and $k = 2$.

32. (D/E) Let $\mathrm{f}(x) = x^n + a_1x^{n-1} + ... + a_{n-1}x + a_n$, where n is a positive integer and the a_i are real numbers. The roots of the equation $\mathrm{f}(x) = 0$ are $-k_1, -k_2, ..., . - k_n$. That is,

$$\mathrm{f}(x) = (x+k_1)(x+k_2)......(x+k_n)$$

By considering f(0) show that $k_1 \times k_2 \times ..... \times k_n = a_n$.

Similarly find $(k_1+1)(k_2+1)......(k_n+1)$ and $(k_1-1)(k_2-1)......(k_n-1)$ in terms of $a_1, a_2, ..., a_n$.

Hence, or otherwise, solve the equation $x^4 + 22x^3 + 172x^2 + 552x + 576 = 0$ given that it has negative integer roots.

33. (C) The cubic equation $250x^3 - 825x^2 + 885x - 308 = 0$ has three real roots that form an arithmetic series. Use the theory of roots of polynomial equations to find the values of the sum and the product of the three roots. Hence find an equation satisfied by the common difference and then find all three roots.

34. (A) Use the factor theorem to show that $(x - 2y + 1)$ and $(x - 3y + 5)$ are factors of the expression

$$6y^3 - y^2 - 21y + 2x^2 + 12x - 4xy + x^2y - 5xy^2 + 10$$

Hence factorise the expression completely.

35. (C) Given two curves $C_1 : \ (x+2)^2 + 2y^2 = 18$ and $C_2 : \ 9(x-1)^2 + 16y^2 = 25$. Identify the intersection points of C_1 and C_2 by eliminating either x^2 or y^2 from the two equations. Hence show that the equation of any circle passing through the intersection points of C_1 and C_2 can be written in the form $x^2 - 2ax + y^2 = 5 - 4a$, where a is some constant.

36. (B/C) A common tangent to two curves is a line that is tangent to the two curves, but not necessarily at the same point.

Find, in terms of a and b, the explicit equation of the common tangent to the two curves $y = x^2 + ax + b$ and $y = x^2 + bx + a$, where $a \neq b$.

Also find, in terms of a and b, the x coordinates of the points of contact of this common tangent with each of the two curves.

37. (A/B) It is given that the shortest distance from the point with coordinates (p, q) to the line $ax + by + c = 0$ is given by

$$d = \frac{|ap + bq + c|}{\sqrt{a^2 + b^2}}$$

Use this formula, geometrical considerations and the fact that $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{1}{\dfrac{\mathrm{d}x}{\mathrm{d}y}}$ to find the shortest distance between the curve $y^2 = 4x$ and the line $y = 2x + 3$.

38. (C/D) Given that x, y, and z satisfy the equations

$$x^2 - yz = a,$$
$$y^2 - zx = b,$$
$$z^2 - xy = c,$$

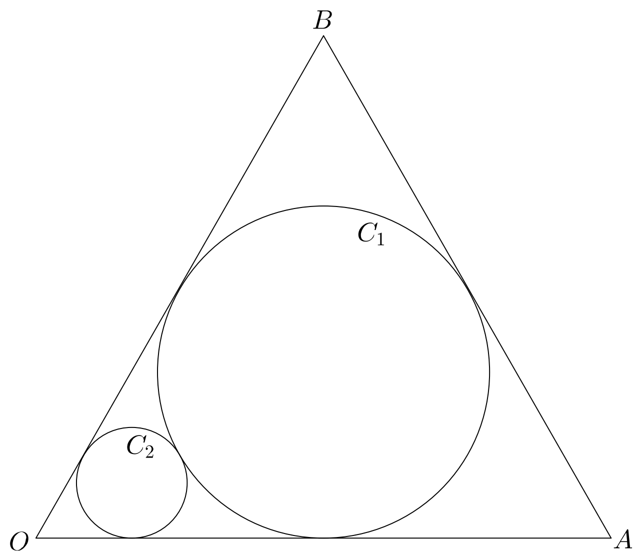
where a, b and c are distinct positive real numbers, show that

(a) $\dfrac{y-z}{b-c} = \dfrac{z-x}{c-a} = \dfrac{x-y}{a-b} = \dfrac{1}{x+y+z}$

(b) $(x-y)^2 + (y-z)^2 + (z-x)^2 = 2(a+b+c)$

(c) $(x+y+z)^2 = \dfrac{a^2+b^2+c^2-bc-ca-ab}{a+b+c}$

39. (A) The diagram below shows a circle C_1 inscribed in an equilateral triangle of side 6 units. C_2 is a circle that touches C_1 and two sides of the triangle as shown. Given that the base of the triangle is on the x axis with the origin O at the lower left vertex.



(a) Find the coordinates of the centres and equations of C_1 and C_2.

(b) What is the equation of the circle C_3 that touches C_2 and the sides OA and OB?

40. (A/B) At the point with coordinates (α, β) the curve with equation $\dfrac{a}{x} + \dfrac{b}{y} = 1$, where $ab \neq 0$, has gradient $-\dfrac{a\beta^2}{b\alpha^2}$.

    It is given that the point with coordinates (p, q) lies on both the line $ax + by = 1$ and the curve $\dfrac{a}{x} + \dfrac{b}{y} = 1$ and at this point they both have the same gradient. Determine the relation between p and q. Hence show that either $(a-b)^2 = 1$ or $(a+b)^2 = 1$

41. (D/E) A sequence is defined by $u_1 = 1,\ u_2 = N$ and $u_n = \dfrac{u_{n-1}^2 + 2}{u_{n-2}}$, for $n \geq 3$.

    Show that $\dfrac{u_{n-1}}{u_{n-2}} = \dfrac{u_n + u_{n-2}}{u_{n-1} + u_{n-3}}$ for $n \geq 3$.

    Hence show that $u_n = au_{n-1} + bu_{n-2}$ for $n \geq 3$, where the values a and b have to be determined.

    For what values of N is each u_n an integer?

42. (D/E) An increasing sequence is defined by $u_1 = u_2 = u_3 = 1$ and

    $$u_{n+1} = \frac{1 + u_n u_{n-1}}{u_{n-2}},$$

    for $n \geq 4$.

    Prove that $u_{n+1} = au_{n-1} + bu_{n-3}$ for $n \geq 3$, where a and b are integers to be found. Deduce that each u_n an integer.

43. (B/C)

    (a) $a_1, a_2, ..., a_n$ is an arithmetic sequence with sum S_n. Find a formula for S_n involving **only** the terms n, a_1 and a_2.

    (b) $b_1, b_2, ..., b_n$ is a sequence such that $(b_2 - b_1)$, $(b_3 - b_2)$, $(b_4 - b_3)$,........,$(b_n - b_{n-1})$ is an arithmetic sequence. Find a formula for b_k, where $k \geq 3$, **only** in terms of k, b_1, b_2 and b_3.

44. (C/D) A triangle has sides of length a, b and c.

    Explain why $a + b \geq c, b + c \geq a, c + a \geq b$.

    Expand $(a + b + c)^2$.

    By considering the positivity of $(a-b)^2 + (b-c)^2 + (c-a)^2$ prove that

    $$3(ab + bc + ca) \leq (a + b + c)^2 \leq 4(ab + bc + ca).$$

45. (B/C). The angle bisector of angle C in triangle ABC meets AB in D.

    Prove that $CD = \dfrac{2ab\cos(C/2)}{a+b}$.

46. (A/B) Show using the addition law for the tangent that if a line with gradient α is reflected in a line whose normal is of gradient ρ to obtain a line with gradient β, then α, β and ρ satisfy

$$2\rho(1-\alpha\beta) = (\alpha+\beta)(1-\rho^2).$$

47. (C/D)

    (a) Show that the family of parabolae with equations $y = x^2 + 2ax + a$, where $a > 1$, have a common point and that their minimum points lie on a parabola.

    (b) For the parabolae with equations $y = x^2 + 2ax + a$, where $a > 1$, that have three intercepts on the axes, determine the equation of the circle that passes through these three intercepts. Hence identify the coordinates of any common point(s) of the family of these circles.

48. (C/D) The family of parabolae with equations $y = x^2+px+q$, where $p^2 > 4q$, have three intercepts on the axes. Show that the equation of the circle that passes through these three intercepts is $x^2+y^2+px-(1+q)y+q = 0$. Hence identify the coordinates of any common point(s) of the family of these circles.

49. (B/C)

    (a) If a and b are positive numbers then show that $\dfrac{a+b}{4} - \dfrac{ab}{a+b}$ is also positive.

    (b) If a, b and c are positive numbers then show that

$$\frac{a+b+c}{2} - \frac{ab}{a+b} - \frac{bc}{b+c} - \frac{ca}{c+a}$$

    is also positive.

50. (B/C) Given two parabolae $y = x^2 + ax + b$ and $y = x^2 + bx + a$, where $a \neq b$.
    Find the equation of the tangent that is common to both parabolae in terms of a and b.
    Determine the x coordinate of the point of contact of the common tangent to each of the two parabolae.