A guide to the use of practical tasks and manipulatives in the teaching of fractions and decimals with children aged 3 to 11

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REVIEW OF RESEARCH

Meanings and models

We have reviewed research and current guidance on teaching fractions and decimals, drawing particularly on research-based guidance. This has helped us to identify common difficulties for children and make recommendations for teaching. Sources include Dickson, Brown and Gibson (1984), Nunes and Bryant (2009), Siegler, Carpenter, Fennell, Geary, Lewis, Okamoto, Thompson and Wray (2010), Lamon (2012), Watson, Jones and Pratt (2013), Rycroft–Smith, Gould and Rushton, (2019), Van de Walle, Karp and Bay-Williams (2021) and Clements and Sarama (2021).

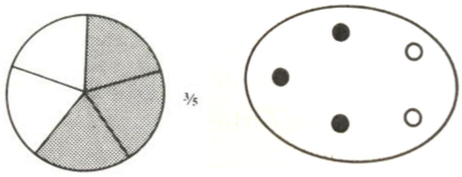
'When we move from whole numbers to fractions, the mathematics takes a giant leap in complexity' says Lamon (2012: 261). Difficulties with learning and teaching often arise because 'fractions' can have several meanings. Much research draws on Kieren's (1976) taxonomy of 5 meanings (e.g. Behr et al., 1997; Charalambous and Pitta-Panzi, 2007; Hodgen, Kuchemann, Brown and Coe, 2010). For instance, $\frac{3}{4}$ can have different meanings:

- parts of a whole e.g. 3 parts of a whole divided into 4 equal parts
- operator e.g. $\frac{1}{4}$ of 3; 3 x $\frac{1}{4}$; $\frac{3}{4}$ of the class
- quotient e.g. the result of 3 items divided by 4
- measure e.g. $\frac{3}{4}$ of a metre or $\frac{3}{4}$ of a kilo
- ratio: e.g. 3:1, the relationship of $\frac{3}{4}$ to $\frac{1}{4}$

The variety of fraction meanings and usages ‘may not be linked in children’s minds’, as Dickson et al. (1984) point out, in a seminal summary of research, which has been subsequently endorsed by much research and guidance. All of these meanings can be complex and present challenges for learning and teaching, raising issues as to how they should be introduced and in what order. For instance, fractions as quotients seem to be less commonly understood e.g. that 3 ÷ 5 results in $\frac{3}{5}$: this is a complex idea, according to Nunes, Bryant, Hurry and Pretzlik (2006). Kerslake (1984) suggests that starting with the part-whole model is not the best approach to develop the underlying idea of fractions as numbers.

Ratio is a much more advanced concept, comparing part to part, which most adolescents find difficult, according to Dickson et al. (1984). Lamon (2012) argues that some young children see part-part situations more easily than part-whole and can access ratios from an early age, so there is no need to defer ratio experiences. While it is generally recommended that children should be taught to connect all these meanings, this is a complex process, requiring multiplicative thinking and later, proportional reasoning. As Lamon also points out, fractions, ratios and other multiplicative ideas are both psychologically and mathematically complex and interconnected, and so it is not possible to specify a linear ordering of topics. The first four meanings seem to be greater priorities for foundational understanding and more relevant to primary age children. Some of the key ideas involved provide a basis for proportional understanding: for instance, unitising, i.e. considering a group of items as ‘the whole’, can help develop multiplicative thinking, and ideas of shares as ‘so many items per person’ link to later ideas of rate and ratio. However, it seems that ratio may be more effectively deferred to the secondary years, when children are more likely to understand ratio in relation to speed, density and gradient, or to visualize scaling with continuous quantities, involving ‘shrinking and stretching’.

Dickson et al. (1984) usefully suggest key models for teaching, separating ‘part-whole’ into two aspects, a ‘sub-area’ of a whole region and a ‘subset’ of a group of objects.



$\frac{3}{5}$

Fig. 1. of an area and of a set (Dickson et al., 1984:274)

The 'area' model can also be used to show division (e.g. with 3 wholes divided by 5) and the set model leads on to the idea of a fraction as an operator (e.g. $\frac{3}{5}$ of 100). They also suggest that the area model can link to the idea of a fraction as 'a sub-length of a unit length' so leading to understanding fractions on a number line and decimal fractions in measurement contexts (1984:281).

Some guidance suggests there are three main contextual models for teaching at primary level (e.g. Lamon, 2012; van de Walle et al., 2021):

- an area model, where the whole is a single object or shape
- a set model, where the whole is a number of objects
- a number line, or length model, where a fraction is a point marking a distance from zero (linking to units on measuring scales and decimal notation).

Dickson et al. (1984) report that, in some studies, children found the number line model more difficult to understand. Number lines involve units that are shown as continuous, and symbols are necessary to make sense of them: children need to integrate visuo-spatial and symbolic understanding, according to Bright, Behr, Post and Wachsmith (1988). It has been generally recommended that teaching activities should begin with fractions as parts of single objects and sets, progressing to parts of a unit of measurement (e.g. Siegler et al., 2010; Clements and Sarama, 2021).

However, there are issues with part-whole approaches. Nunes et al. (2006) report that young children found dividing a single object or shape harder than sharing groups of objects. Working out the size of the whole from a fractional part can be easier to visualize with a circle than a rectangle. However, rectangles are useful for showing multiplication: for instance $\frac{1}{2}$ x $\frac{2}{3}$ can easily be seen by cutting a rectangle in two directions. Lamon (2012) recommends hybrid models where a whole item is marked into sections, like a chocolate bar, or several items are arranged together, like eggs in a box or a pack of yoghurt pots.

Researchers agree that introducing fractions in association with a part-whole model can result in this meaning dominating and inhibiting the development of more complex and abstract fraction concepts (e.g. Hodgen et al., 2010; Education Endowment Foundation, 2017). Approaches focusing on a single whole, whether a shape or set, may tend to ignore fractions bigger than one, as Dickson et al. (1984) point out, whereas the number line includes mixed numbers. This helps children to see fractions more abstractly, as numbers ordered between other numbers and as decimals. More recent guidance recommends emphasizing the number line model, and so drawing attention to the ordinal aspects of fractions, particularly with older primary children (Siegler et al., 2010; EEF, 2017.)

In order to connect the different fraction meanings, children need to connect the different models. One approach, as suggested by Dickson et al. (1984), is to treat fractions of a metre as parts of a whole, and then relate these to fractions on a measuring tape. Folding paper strips is frequently recommended as an introduction to a length model e.g. in the USA, (Siegler et al., 2010) and Japan (Trundley and Burke, 2020). Strips can be lined up to show equivalent fractions as 'fraction walls'. English guidance has suggested using bar models instead of set models to represent division problems, thereby linking part-whole understanding to length models (Department for Education, 2020).

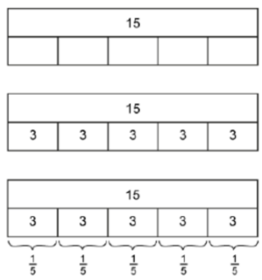


Fig 2. Using a bar model to show fractions (DfE, 2020:125)

Alternatively, a Dutch approach combines a set model with a length model by using beadstrings in first grade, which can then be linked in progression to number lines, measuring scales and fraction/percentage bar models (van den Heuvel-Panhuizen, 2000).

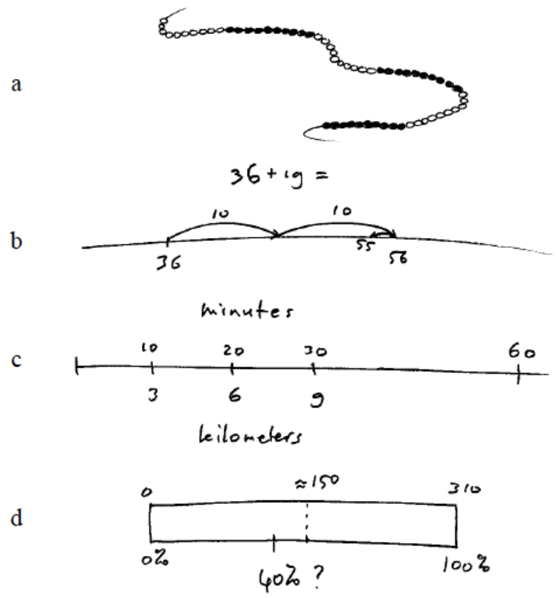


Fig 3. A progression in use of models (van den Heuvel-Panhuizen, 2000:7)

Key manipulatives that provide a length model for teaching fractions are Cuisenaire rods, promoted by Gattegno (1957). The coloured rods are proportional to one another, making them a powerful resource to represent fractions. Goutard (1964), following Gattegno, describes investigative activities such as finding pairs of rods with the same proportional relationship. She found that young children were able to reason about fractions, using proportionality: her examples of their work are very interesting.

Fig. 4. Cuisenaire rods showing $\frac{2}{3}$, $\frac{4}{6}$, $\frac{6}{9}$

Key ideas

Research literature identifies a number of key issues for learning and teaching fractions, that have implications for teaching or that might give primary age children difficulty.

Fractions are equal parts

A key idea is that fractions are not just parts of a whole, but have to be equal e.g. cutting an apple into two parts may not result in two halves. While fractions of a single object or shape seem an accessible model for young children, in fact it can be hard to judge the equality of parts within a whole and young children lack the fine motor control to cut or fold precisely (Nunes & Bryant, 2009). Equal parts can be easier to identify when sharing a number of objects, so this may be a more effective context to use first. However, Nunes and Bryant also point out that young children may readily 'deal' objects one-to-one when sharing, without realizing that equal shares should result, so they need to discuss the fairness of resulting shares. Of course, finding a fraction of a set of items by dividing their number only works if the items are the same size or value (Lamon, 2012). For instance, dividing a number of different toys or sweets into equal numbers may not result in shares that are considered fair. This issue is ignored by examples showing groups of unrealistically identical objects, suggesting that more realistic situations need discussion too.

When something is split into equal parts, these do not have to be the same shape, or in the same arrangement or orientation (Charalambous & Pitta-Panzi, 2007). A common misconception is that equal parts of a whole must be the same shape, rather than have the same area or quantity. Therefore teaching needs to present equal fractions that vary in appearance in different ways.

Naming fractions

The names of fractions, such as a half, are idiosyncratic in many languages. There are, of course, differences in English usage e.g. quarters in UK are fourths in USA. The pattern of words for fractions becomes more predictable from a sixth. Fraction names may be confusing because they sound like other numbers (e.g. hundreds and hundredths) or they have alternative meanings (e.g. coming third in a race). Van de Walle et al. (2021) and others report that children may also confuse the number of parts in a fraction with the number of parts in a whole e.g. calling $\frac{3}{4}$ 'thirds'.

Comparing fractions

A novel feature of fractions for young children is that the larger the denominator, the smaller they are, rather than larger numbers representing a greater quantity. However, Nunes et al. (2006) found that very young children can understand the inverse relation between the number of people sharing and portion size, leading to the understanding that the more parts something is divided into, the smaller the parts are. Sharing is therefore an important early experience and context to build on. Suggestions about delaying the teaching of fractions fail to take account of the importance of informal knowledge as the basis for learning, according to Nunes and Bryant (2009). With older children, number lines 'help children see fractions as numbers that can be compared- an important but often overlooked concept and skill', according to Clements and Sarama (2021:159). Similarly, Siegler and Braithwaite (2017) found the number line was more effective to teach comparison to 9 and 10 year olds, than counting fractions on pictures (or a part-whole model).

Relativity to the whole

The same fraction may represent different amounts, if the wholes are different e.g. half of small cake is smaller than half a large one. Fosnot and Dolk (2002) point out that in order to compare two fractions the whole must be the same: the whole matters because fractions are relative amounts. Children may find it hard to understand this fundamentally different nature of fractions from the whole numbers they have met previously.

Lamon (2012:98) states that in every new context the first question children should always ask themselves is *'What is the unit?'* She suggests that many children have not developed fraction sense, because they never grasped the importance of the whole. Unitising, or being able to think of a group of items as one item, is a key development in mathematical thinking for children. This is essential for the multiplicative reasoning required for a deep understanding of fractions. It leads to being able to consider one item as part of a composite unit and of other alternative units within it. Spatial visualization skills, including mentally manipulating images, are crucial in this process (Cutting, 2019).

Fraction symbols: fractions as numbers

Fractions combine numerals to give a composite meaning where the 'top number' is proportional to the 'bottom number' (Van de Walle et al., 2021). In different contexts these symbols can mean different things, for instance $\frac{2}{3}$ can mean 2 out of 3 equal parts, or a number on a number line between $\frac{1}{2}$ and 1. Children's understanding of practical problems involving fractions is often ahead of their knowledge of fraction symbols (Nunes & Bryant, 2009). Clements and Sarama (2021:159) recommend 'math-talk-rich experiences, with careful introduction of symbols after verbal language has been established'. Even older children may think of the digits as whole numbers: for instance, when comparing $\frac{2}{3}$ and $\frac{2}{5}$, they may see the 3 in $\frac{2}{3}$ as smaller than the 5 in $\frac{2}{5}$. Gilmore, Göbel and Inglis (2018) report that up to the age of about 14, many children assume fractions behave like whole numbers, and that in one study only 37% of 11 to 12 year olds knew $\frac{4}{5}$ was larger than $\frac{4}{7}$. Teachers may also not understand, for example, that multiplication by a fraction less than one results in a smaller number. Even mathematicians have to inhibit 'whole number bias'. Studies suggest that focusing on numerical magnitude by placing numbers on a number line reduces this.

The relationship with division: fractions as quotients

The link between the numbers involved in division and the digits in the resulting fraction (e.g. $2 \div 3 = \frac{2}{3}$ and $\frac{2}{3} = 2 \div 3$) was not apparent to 66% of 12 and 13 year olds in a study reported by Dickson et al. (1984). This is a complex idea even in its simplest form, as pointed out by Nunes et al. (2006). For instance, with one item shared between four, the fraction $\frac{1}{4}$ indicates both the division (1 divided by 4) and the portion that each one receives (the quotient, or result of division). They suggest that sharing provides opportunities to explore fractions used in this way. Flexibility of part-whole thinking is required to understand that, for instance, one third of 2 matching cookies is the same size as $\frac{2}{3}$ of one cookie. This leads to the idea of the commutativity of fractions, that $\frac{1}{3}$ x $\frac{2}{1}$ = $\frac{2}{3}$ x $\frac{1}{1}$ (symbolically, this also involves understanding that whole numbers can be expressed as fractions too.)

Equivalent fractions

The same quantity can have many different fraction names.

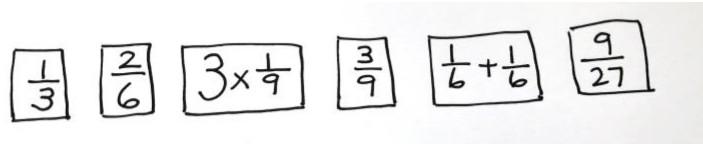


Fig. 5. Example of fractions equivalent to a third

Practical sharing problems can usefully show that different methods result in equivalent fractions e.g. 3 items can be shared between 4 by giving each person one quarter from each item (3 separate quarters) or giving them a half and a quarter (making three quarters altogether) or giving 3 people three quarters and the fourth person separate quarters (Nunes et al., 2006). A related but more advanced idea is co-variance: if both the top and bottom numbers in a fraction are multiplied or divided by the same number, the size of the fraction is the same relative amount. Young children can understand that if there are twice as many things to share and twice as many people involved, then each share will be the same, according to Nunes and Bryant (2009). This implies that it is important to offer children a variety of sharing problems.

Fractions on a number line

The number line is a key model for understanding fractions as abstract and composite numbers, but substantial work with whole numbers on a number line will need to come first (Murphy, 2011). Placing numbers on an empty number line at age 6 predicts rational number understanding at 13, according to Siegler and Braithwaite (2017). As mentioned previously, they also report that number lines provide a more effective way of teaching the relative value of fractions with 9 to 10 year olds, than using part-whole models.

Charalambous and Pitta-Panzi (2007:300) suggest that 'assigning the concept of numberhood to fractions is a leap': when children are not familiar with a counting sequence including fractions, they may expect $\frac{2}{3}$ to be between 2 and 3. The number line can also show that there are an infinite number of fractions between two fractions, the idea of density of fractions. However, most 14 and 15 year olds do not understand this, according to Gilmore et al. (2018).

Much research and guidance follows Dickson et al.'s (1984) recommendation to introduce the number line through practical measurement, so units have clear meanings (e.g. Rycroft-Smith et al., 2019, Siegler et al., 2010). Using measurement as an approach to understanding number lines may also help children to avoid the common error of counting marks instead of intervals. As measuring in the UK is usually in metric units, it is important for children to have practice with using lines marked in decimals, according to Watson et al. (2013:71).

Mixed numbers
These tend not to be included in examples focused on a single whole unit, as they are difficult to show diagrammatically for areas and numbers (Dickson et al., 1984). Mixed numbers tend to be only shown on number lines. English guidance from the National Centre for Excellence in the Teaching of Mathematics (NCETM, 2019) puts these together, using an image of improper fractions and mixed numbers on a number line, combined with pictorial representations of orange quarters.

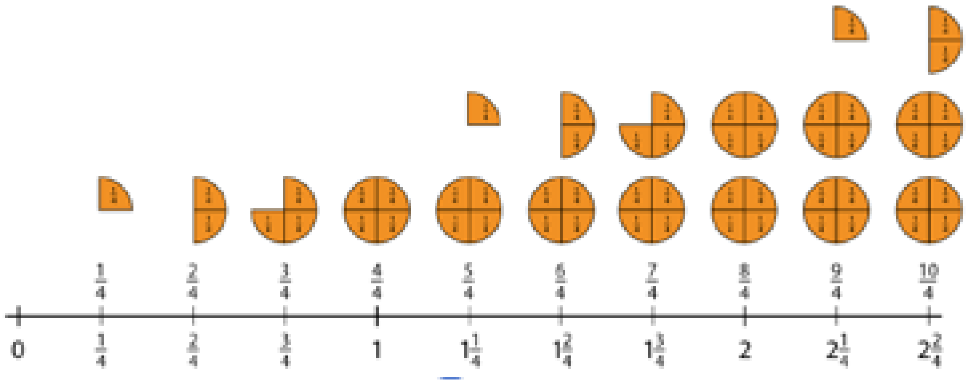


Fig. 6. Combining part-whole and number line models
(NCETM, 2019 Spine 3 05.3 Y4)

Experiences with mixed numbers also provide opportunities for children to understand whole numbers as fractions e.g. $\frac{2}{1}$, $\frac{4}{2}$, $\frac{4}{4}$.

Decimal fractions
It is interesting that Dickson et al. (1984:284) predicted nearly forty years ago that decimals would dominate and 'the use of fractions at other than an informal level will die out': however, current curricula do not entirely show this. Some researchers argue that decimal fractions are easier to compare (e.g. Siegler & Braithwaite, 2017): however, decimal notation requires an understanding of the significance of the decimal point and fluency with place value. As Dickson et al. (1984) point out, understanding the digits after the decimal point depends on fraction concepts of 'tenth' and 'hundredth'. They suggest that relating decimals to more concrete meanings in the context of measurement, and also using calculators, should prevent children's difficulties with decimal notation; the former rather than the latter seems to be reflected in current guidance.

Contexts for learning
The literature is generally agreed on the need to avoid introducing fractions in

the abstract, and to provide activities and experiences that show different meanings, embed key ideas and avoid misconceptions. Two main contexts for activities emerge in primary school: sharing and measurement (Nunes & Bryant, 2009; Rycroft-Smith et al., 2019).

Sharing

Research suggests the importance of drawing on early experiences of sharing. Clements and Sarama (2021) suggest building on young children's intuitive ideas of 'fairness' to check whether something is split in half. However, Hunting (1999) warns that children's understanding of fairness may depend on their home experiences, which may vary considerably: for instance in some families, shares may vary according to age and not be equal, while some children may not experience sharing at all. This implies that children need such experiences in schools and early childhood settings.

Nunes et al. (2006) recommend using a variety of situations and revisiting fractions at different levels of difficulty. With older children, they found that 9 to 10 year olds were more successful with social sharing problems than shading parts of a shape e.g. to find $\frac{2}{3}$ of 6 or 9 portions. They conclude that focusing on visual partitioning distracts children from thinking logically about equality resulting from division and that relating to experiences of social sharing encourages reasoning with understanding.

Sharing contexts have been recommended for helping children understand relativity to the whole, the inverse relation between denominator and quantity, the relationship with division and equivalence including co-variance (Dickson et al., 1984, Nunes & Bryant, 2009, Van de Walle et al., 2021). This implies the need for children to experience a range of scenarios, including sharing a number smaller than the divisor, such as 2 shared between 3, or with a remainder, such as 7 shared between 3, and also varying the numbers of items and people within one scenario. Comparing different ways of sharing stimulates children to discuss and draw diagrams that promote reasoning and understanding of equivalence (Nunes et al., 2006).

Practical sharing problems also reveal children's understanding: for instance, Lewis, Gibbons, Kazemi and Lind (2015) analyse 5 different strategies used by 9 and 10 year olds for sharing 8 sandwiches between 6 children, including 'non anticipatory' halving, and dividing each item. A rich example of a contextualised problem involving ideas of equivalence and comparison is Fosnot and Dolk's (2002) case study of older primary children sharing 'submarine sandwiches'. This presents a length model to help comparison and show equivalence, stimulating children to draw their own diagrams to show their thinking. As with younger children, the issue of fairness provides strong motivation to engage with a problem about sharing.

Measurement

Researchers agree that the number line is important for enabling comparison, showing fractions as numbers in between whole numbers and as decimals (e.g. EEF, 2017; Gilmore et al, 2018; Rycroft-Smith et al, 2019). Measurement activities allow children to engage with number lines practically and give meaning to units, as pointed out by Dickson et al. (1984). Siegler et al. (2010) recommend that teachers emphasise the way that fractions allow for more precise measurement of quantities. According to Lamon (2012), this is a key idea: units of measure can be divided up into increasingly small units to make them as accurate as you want. She argues that introducing fractions through measurement gives children a greater understanding of relative sizes, relative positions on a number line and equivalence, helping them to develop flexible 'fraction sense'. It also helps children to interpret number lines marked at different intervals and those with unlabelled marks. Watson et al. (2013:71) suggest that measurement, especially using decimals, 'needs repositioning as a key component of mathematics learning in the 9 to 19 age range'. Strikingly, the Japanese approach to teaching fractions focuses mainly on measurement contexts (including capacity as well as length) and almost to the exclusion of part-whole models (Trundley & Burke, 2020). Measurement is increasingly emphasised in current curricula revisions, e.g. Australian Curriculum (2022) and USA Common Core Standards (2022).

Approaches to learning fractions and decimals

The research into children learning about fractions emphasises the need for activities such as practical problem-solving, which involve children in discussing varied situations, drawing diagrams, visualizing and generalizing from multiple, carefully chosen examples. Brown (2022) points out that fraction use in everyday life is much reduced, but their use in comparison and ratio problems is probably more important than ever. This means that ideas of equivalence and multiplication of fractions are extremely useful, especially for older learners.

Research warns about children's difficulties in understanding abstract fractions and the dangers of teaching procedures without understanding (e.g. Siegler & Braithwaite, 2017). This raises broader issues about the nature of mathematical learning and echoes Clements and Sarama's view, acknowledging Schoenfeld (2021:287) that 'children must see all maths as a search for patterns, structures and relationships, as a process of making and testing ideas, and in general, making sense of quantitative and spatial situations'. Kilpatrick, Swafford and Findell (2001) also stress that mathematical proficiency is complex, and propose a model of five strands described as productive disposition, strategic competence, adaptive reasoning, conceptual understanding and procedural fluency.

One implication is that activities must engage children positively and productively, while providing extended opportunities to develop fluency and understanding. Multiple examples are needed to engage children in *manipulating, getting a sense of* and *articulating* key ideas (Mason & Johnston-Wilder, 2004). It is worth noting that calculators provide a useful way of generating examples from which children can spot patterns and generalise. When exploring decimal fractions, evidence from the 1980s onwards shows that children taught through a 'calculator aware' approach made greater use of mental methods, which is key to working fluently with decimals (Ruthven, 1998). The calculator is therefore an important tool to help children understand fractions and decimals as abstract numbers.

As Fosnot and Dolk (2002) exemplify in their work, engaging contexts can provide children with extended opportunities to talk and think about different examples, to explain and to experiment, while providing useful representations. Their approach is related to 'Realistic Mathematics Education' in the Netherlands, which, according to Treffers and Beishuizen (1999), uses contexts where children can 'reinvent' mathematics by using relevant representations to solve problems.

Lamon (2012) advocates a similar approach from her research, involving discussion and reasoning about challenging problems, using varied models, set in a range of real-world contexts so children can draw on their experience and intuitive understandings.

Problems and investigations provide opportunities for reasoning and the development of language and vocabulary, visualization and children's own graphical representations of their methods and solutions. However, many examples and a considerable amount of time are needed for secure understanding and for 'fraction sense' to develop.

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