A guide to the use of practical tasks and manipulatives in the teaching of fractions and decimals with children aged 3 to 11

MAIN REPORT

Rose Griffiths, Jenni Back and Sue Gifford
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INTRODUCTION

Children who are confident with counting and making sense of whole numbers will discover that there are some problems that need a new kind of number, a fraction, to solve them. However, fractions and decimals have long been recognised as difficult to teach.

We have previously developed guidance (Griffiths, Back & Gifford, 2017) about practical ways to teach arithmetic with whole numbers. The Nuffield funded project *'Using manipulatives in the foundations of arithmetic'* recognised that practical experience and real-life situations help children to make sense of whole numbers. It was clear that fractions would benefit from a similar approach.

We recognise that fractions and decimals are interconnected and that both are fractions i.e. rational numbers. We will generally use 'fractions' to mean any number that can be written with a 'top' and a 'bottom' number, like ¾, 2 ½ and 22/5, and 'decimals' to mean numbers like 3.14, 6.0 and 0.782.

Previous research (Watson, Jones & Pratt, 2013) has shown that pupils in secondary schools lack fluency with fractions and decimals, indicating that more needs to be done in primary schools to strengthen foundational knowledge. Primary school teachers who lack confidence in their own subject knowledge may struggle to ensure that children develop a sound conceptual understanding. We aimed to create a progression of activities that would give teachers a broader repertoire of interesting ways to improve children's understanding and fluency.

Consequently our main research question was:

- What exemplars would be useful to teachers to build their understanding and knowledge of effective approaches to teaching fractions and decimals, and, where needed, to improve their own subject knowledge?

In order to answer this we asked:

- What existing research evidence is there about the teaching and learning of fractions and decimals with children aged 3 to 11?

- What guidance has been provided to primary teachers on these topics, both in the pre-decimalisation and pre-calculator period (before 1975) and since then?

- What is currently common practice in teaching about fractions and decimals in English primary schools?

The two most meaningful contexts, highlighted in our review of research about teaching fractions and decimals, were social sharing and purposeful measuring. These contexts engage children in making sense of fractions through practical activity, discussion and informal recording, before moving towards formal representations.

To complement the major support provided by the Nuffield Foundation, we worked with Oxford University Press to publish the exemplars in a book for teachers, *Making Fractions,* accompanied by free online materials. There are three publications from this project available on the Nuffield Foundation website: Main Report, Review of Research and Examples for Teachers.

REVIEW OF RESEARCH AND GUIDANCE

Initially we drew on seminal research and research-based guidance to identify recommendations for teaching. Sources included Dickson, Brown and Gibson (1984), Nunes and Bryant (2009), Siegler, Carpenter, Fennell, Geary, Lewis, Okamoto, Thompson and Wray (2010), Lamon (2012), Watson, Jones and Pratt (2013), Van de Walle, Karp and Bay-Williams (2020) and Clements and Sarama (2021). A fuller review of the research surveyed can be found in our separate Review of Research.

Fractions are tricky because they are complex. Much research and guidance uses Kieren's (1976) taxonomy of 5 meanings. For instance, $\frac{3}{4}$ can have different meanings:

- parts of a whole e.g. 3 parts of a whole divided into 4 equal parts
- operator e.g. $\frac{1}{4}$ of 3; 3 x $\frac{1}{4}$; $\frac{3}{4}$ of the class
- quotient e.g. the result of 3 items divided by 4
- measure e.g. $\frac{3}{4}$ of a metre or $\frac{3}{4}$ of a kilo
- ratio: e.g. 3:1, the relationship of $\frac{3}{4}$ to $\frac{1}{4}$

Dickson et al. (1984) usefully separate 'part-whole' into two aspects, a sub-area of a whole region and a subset of a set of objects. A sub-area of a whole region links to ideas of fractions as shaded areas of shapes. The notion of a subset of objects links to seeing fractions as operators and as quotients (e.g. $\frac{3}{4}$ resulting from 3 divided by 4). Measurement leads to an understanding of fractions as points on number lines. Ratio is a more advanced concept, involving relationships between parts. Sources agree on the importance of seeing fractions as interrelated concepts and suggest three main contextual models for teaching:

- wholes as regions (or shapes)
- wholes as a number of objects
- the number line (linked to measurement)

An underpinning idea in all of these is that a fraction is always a fraction of 'something' and the 'whole' affects the value of the fraction. When fractions are on a number line, the whole is the number one. Some guidance suggests a teaching sequence beginning with dividing or sharing, progressing to measurement and number lines, with ratio as a later development.

In the literature several key ideas are identified which primary age children may find difficult, giving rise to possible misconceptions. These key ideas provide an important teaching focus and include:

- each fraction is made by dividing something into equal parts
- equal parts may appear different
- the more equal parts, the smaller they are
- equivalent fractions have many names.

Sharing
Nunes, Bryant, Hurry and Pretzlik (2006) found that drawing on children's early experience of social sharing, rather than dividing shapes, allows children to reason more effectively about fractions. Experiences of sharing also help children understand key ideas, such as relativity to the whole and the inverse relationship between the number of parts and the size of each part. Different methods of sharing can also demonstrate equivalence.

As Fosnot and Dolk (2002) exemplify in their work, engaging contexts can provide children with extended opportunities to talk and think, explain and experiment, while providing useful representations. This approach is related to 'Realistic Mathematics Education' in the Netherlands, which, according to Treffers and Beishuizen (1999), uses contexts where children can 'reinvent' mathematics by using relevant representations to solve problems.

Measurement
Watson et al. (2013:71) suggest that measurement, especially using decimals, 'needs repositioning as a key component of mathematics learning in the 9 – 19 age range'. Practical measurement activities can use decimals and exemplify the number line. This is important for showing fractions as numbers between whole numbers and encouraging comparison.

Working with others to measure distances encourages children to check their measurements and find ways of making them more accurate. Barclay (2021) found that high-attaining and low-attaining pupils can benefit from working together, with all pupils making useful contributions to paired activity.

Number sense
Teachers need to help children to build on their number sense (Sayers and Andrews, 2015). In *Making Numbers* (Griffiths, Back and Gifford, 2016) we identified three key ideas in number sense for whole numbers: counting, comparison and composition. They are also useful in considering fractions and decimals. Counting enables children to see that fractions are not always smaller than one and include mixed numbers. Comparison helps children see fractions as numbers that can be compared and put in order on a number line. Composition explores different ways a fraction can be made, for example a third can be three ninths, or one sixth and two twelfths.

The nature of learning mathematics
Existing research and guidance warns heavily about the difficulties children have in understanding fractions as abstract numbers and the dangers of teaching procedures without understanding. Clements and Sarama (2021:287, acknowledging Schoenfeld, 2008) suggest that '*children must see all maths as a search for patterns, structures and*

relationships, as a process of making and testing ideas, and in general, making sense of quantitative and spatial situations'.

Kilpatrick, Swafford and Findell (2001) describe five strands of mathematical proficiency. Children's '*productive disposition',* or motivation to learn something new and to try things out for themselves, is fostered by having interesting problems to tackle. *Strategic competence* and *adaptive reasoning* benefit from discussion of alternative methods and having to explain the decisions you make. *Conceptual understanding* needs to connect a range of representations and examples, as well as needing time to develop; *procedural fluency* will follow.

Developing mathematical thinking

Our review of research emphasised the need for practical problem-solving and investigations. Engaging children with multiple examples, offering opportunities for *manipulating, getting a sense of* and *articulating* key ideas are an important part of this. (Mason & Johnston-Wilder, 2004). This may involve using story contexts, games and pattern making.

Neuroscientific research supports these approaches, suggesting that children construct knowledge through physical interaction with objects and the environment, together with language, imagination and pretend play, and teaching (Goswami & Bryant, 2010). It is important to make links between a range of 3D, 2D and symbolic representations of fractions, talking about what they show, drawing and using informal and standard notation.

Sometimes a specific piece of equipment acts as a catalyst for developing knowledge about fractions. The calculator is an important tool to help children understand decimals. It was found that children taught through a 'calculator aware' approach made greater use of mental computation (Ruthven, 1998). When exploring decimal fractions, evidence from the 1980s onwards showed that calculators can be used to generate examples from which children can spot patterns and generalise.

Many researchers have examined the links between practical activity and symbolic representation. Bruner (1966) refers to enactive, iconic and symbolic modes of representation. Mason interprets these as three different worlds of experience, moving between manipulable objects, drawing or mental imagery and abstract symbols (see Mason, Burton & Stacey, 2010). Children are encouraged to talk about what they notice in their own words and to record in their own ways. This provides valuable insights into children's thinking and understanding for teachers, and also for the children themselves.

The challenge for teachers is in helping children to connect different contexts, models and meanings. Children's understanding may be context specific, according to Nunes et al. (2006). Teachers need to explicitly help children make connections between representations in order to generalise (Cambridge Mathematics, 2019).

Many researchers stress the importance of metacognition, or thinking about thinking. As Askew (2012) notes, a key aspect for teachers is an interest in children's own methods and in sharing other methods with them. The Education Endowment Foundation (EEF, 2017:7) recommends that teachers 'provide regular opportunities for pupils to develop metacognition by encouraging them to explain their thinking to themselves and others', and also to develop pupils' independence and motivation.

International approaches to fractions and decimals

Currently, various jurisdictions are revising curricula and guidance for teaching fractions both in the models used and the age of introduction. For example in the USA and Australia some aspects of fractions are being deferred to later age groups. Recent English non-statutory guidance (DfE, 2020) recommends deferring fractions until children are in Year 3 (7 and 8 year olds). Fractions are introduced with the part-whole model, linked to division, and also the number line, including addition of fractions. In Japan, the number line model is prioritised over the part-whole model, with 7 and 8 year olds folding paper strips and 9 and 10 year olds using measuring scales, showing fractions as part of a whole metre or litre (Trundley & Burke, 2020). In the Netherlands, the use of beadstrings from ages four or five is later linked to number lines and bar models (van den Heuvel Panhuizen, 2000).

US guidance from the What Works Clearing House (Siegler et al, 2010) emphasises sharing as the key introductory experience for fractions, and generally using contexts which help children make sense, particularly regarding calculations with fractions. Similarly, the Netherlands recommends that primary children do not engage in abstract calculations with fractions. Australia and New Zealand emphasise 'everyday fractions', 'common uses' and 'familiar contexts', suggesting the importance of relevant contexts in giving meaning to fractions.

Past approaches and current influences in the UK

We examined a sample of teacher guidance and pupil textbooks from 1900 to the present day. This gave us many things to consider – particularly around two pivotal changes in the 1970s.

- The metrication of measurement and then decimalisation of the UK currency in 1971 changed the balance of teaching time spent on fractions compared to decimals (HMI, 1979). We explored materials that pre-dated 1971 for potentially useful ideas that might have fallen out of use.
- The increased access to affordable electronic calculators from about 1975 led to several projects that developed materials showing how calculators could be used

effectively to explore mathematical ideas, including crucial work on decimals. We added an item to our survey about current practice (see page 10) on teachers' use of calculators when they are introducing decimal fractions.

We have used many of the ideas that follow in *Making Fractions* - especially valuing the focus on meaningful contexts that help children make sense of fractions. These older publications particularly emphasised measuring as a helpful context, alongside sharing, and gave us ideas of ways to include this in an accessible way.

The Royal Society (1969:16) *Metric units in primary schools.*

We drew on the Mathematical Association archive of classroom-based material, which is held in the University of Leicester library, on our own collections of texts, on current texts used in schools and on web-based materials for current guidance. Reviewing past resources confirmed some ideas as longstanding, and we found some valuable ideas which seem to be missing from current guidance. Interpreting this evidence in conjunction with the research reviewed and our knowledge of children's experience with fractions and decimals outside school, we have identified the following key ideas.

Use contexts where children can see fractions are useful

Many texts stressed the importance of making links with fractions that are familiar to children, and of using them in a practical context. For example Sawyer (1964: 312) said: 'Fractions are supposed to be difficult to learn. The reason for the difficulty is almost certainly a wrong approach in teaching. Too often the attempt is made to teach children a variety of complicated procedures without any real understanding, instead of teaching children to see the situation and decide for themselves what procedure is most reasonable.'

The Nuffield Mathematics Project (1968: 70) suggested we should concentrate on fractions 'which the child uses almost every day. There does not seem to be any point in using fractions other than $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, perhaps $\frac{1}{16}$, and possibly $\frac{1}{3}$ because of the school milk-bottle. Certainly there would seem to be no need to introduce 'top-heavy' or 'improper' fractions such as $\frac{8}{5}$ and $\frac{16}{3}$, which do not arise naturally in the normal course of events.'

Thompson (1917:121) talks about imperial measure and fractions: '… the idea and notation of fractions should gradually be introduced in connection with practical work. … halves and quarters of an hour arise in learning to read the clock; halves, quarters and eighths of yards, pounds and quarts in shopping; halves, quarters, eighths and even sixteenths of an inch in connections with measurements for handwork and drawing of plans.'

The literature review and our own experience had already highlighted the importance of measurement as a useful context alongside sharing when learning about fractions. It was useful to be reminded of the many places that children encounter fractions, particularly halves and quarters.

Fraction Families

The idea of a 'fraction family' has been defined differently by different authors: Latham and Trulove (1981) described a fraction family as being a group of equivalent fractions: for example, $\frac{1}{4}$, $\frac{3}{12}$, $\frac{4}{16}$, whereas the Mathematical Association (1956) used a broader definition, so that the family of $\frac{1}{4}$ also included $\frac{2}{4}$, $\frac{3}{4}$ and $\frac{4}{4}$ and fractions equivalent to them. We experimented with both ideas, and decided that a 'family' that encompassed a wider group of fractions was more useful today. It enabled us to help children think about how halves, quarters and eighths can be combined, and how easily thirds and sixths can be made from each other, or fifths and tenths.

Mixed numbers and the number line

When common fractions are first introduced, there has often been an initial emphasis on fractions smaller than one, using a circle or a rectangle as the whole one, with mixed numbers left until later in the primary years.

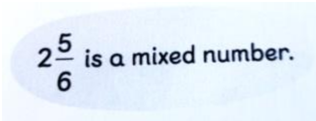
In the past mixed numbers arose more naturally as children used their rulers marked in fractions of inches (as a kind of number line). Whitwell and Goddard (1935: 25) had a section on 'Vulgar Fractions' (an old term for common fractions) which started with just one question that made sure the children could see $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$ and $\frac{1}{10}$ of an inch on their rulers. The next ten pages of exercises used the length of straight lines to explore addition, subtraction, multiplication and division with mixed numbers.

(3) Work these from your rulers—
(*a*) $\frac{1}{2} \div 2$; (*b*) $\frac{3}{8} \div 3$; (*c*) $\frac{3}{4} \div 2$; (*d*) $\frac{1}{4} \div 4$
(*a*) $\frac{1}{2} \div 5$; (*b*) $\frac{1}{4} \div 3$; (*c*) $\frac{2}{3} \div 4$; (*d*) $\frac{3}{4} \div 4$

(4) (*a*) Draw a line $1\frac{1}{2}$ inches long. Mark it off in 8ths.
How many times is $\frac{3}{8}$ inch contained in $1\frac{1}{2}$ inches ? ($1\frac{1}{2} \div \frac{3}{8}$)

We decided to introduce mixed numbers through measurement and sharing in *Making Fractions*. Hargreaves (1982: 113)) used an example of counting in fifths on a number line to show the links between mixed numbers and 'improper fractions' (i.e. 'top-heavy fractions' such as $\frac{6}{5}$.

Currently *Maths-no problem!* textbooks (Singapore Maths, 2016) present a useful and logical way of approaching fractions, but we were disappointed at the way in which mixed numbers are written, with the digits all the same size:



Our preference is for mixed numbers where the fractional part is about the same height in total as the whole number part e.g. $2\,\frac{5}{6}$. The smaller numerals for the fractional part emphasise the fact that this part is smaller than a whole one. This layout is also more suitable for later work in algebra.

Fraction vocabulary

Our sample texts varied in whether they used terms such as denominator, numerator and improper fraction. We decided to follow the pattern in SMP (Schools Mathematics Project, 1970) and use 'top number' and 'bottom number' as being more straightforward, but also referenced the alternative names. Similarly, we felt that 'top-heavy' fraction provides instant understanding of what is meant.

Hargreaves (1982) notes how useful it is to talk out loud about adding and subtracting fractions with the same bottom number. For example, writing '3 fifths' and saying it out loud, rather than just silently reading the symbolic form $\frac{3}{5}$, should make it more obvious that 3 fifths add another 3 fifths must be 6 fifths. It could avoid the misconception that

$$\frac{3}{5} + \frac{3}{5} = \frac{6}{10}$$

where the child has just added top and bottom numbers. Asking questions like 'How many fifths have you got?' is also helpful.

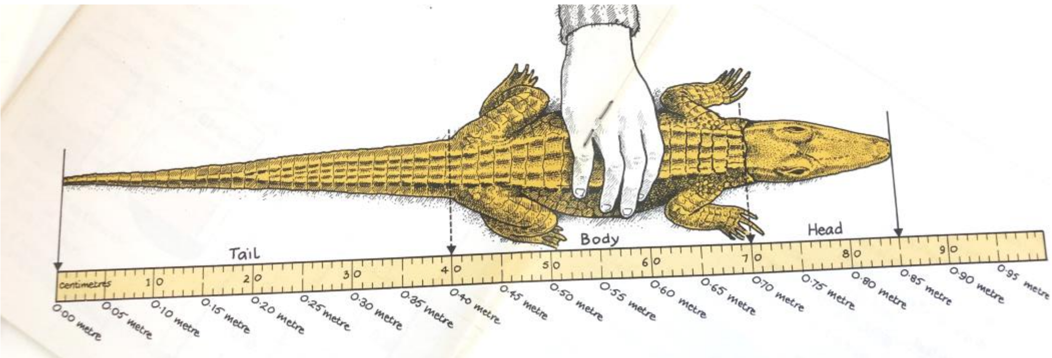
Working with decimal fractions

We could see that many texts kept common fractions and decimal fractions relatively separate. We were concerned to link the two types of fraction more carefully. Using the 'fraction family' idea allowed us to move between halves, tenths and hundredths at a sensible pace.

The Royal Society (1969: 15) presented an optimistic view of children's future confidence with decimals:

> 'The coming of decimal currency and the adoption of metric units will provide such a wealth of practical experience in expressing quantities on a decimal scale that a new understanding of decimal fractions should become general.'

Some texts confirmed that using the measurement of length was a useful context to introduce work with decimal numbers. The School Mathematics Project (SMP, 1983: 8 & 9) provided an engaging example, showing that we have a choice when we want to measure more accurately: to use smaller units, or to use a decimal part of a larger unit.



SURVEY OF CURRENT PRACTICE

We used a questionnaire to gather data from teachers working with children aged 3 to 11 about how they approached teaching fractions and decimals. The initial distribution of questionnaires in early March 2020 was not as successful as hoped, because it reached schools in the same week as the first Covid lockdown. We tried again twice more, achieving an opportunistic sample gathered from across the Midlands, London and the South East, with data from 126 teachers from 19 schools.

Our sample covered 11 urban, 3 suburban and 5 small rural schools with a mixture of levels of deprivation but with all schools classified as good or outstanding by Ofsted. The 126 teachers who responded included 27 working with EYFS, 35 in Key Stage 1, 50 in Key Stage 2 and 14 teachers working with children from a mixture of phases.

Due to the pressures on schools at that time, we kept the questionnaire short. Based on our reading of relevant research and guidance, we decided to ask just three main questions. The first two explored the models and contexts teachers used:

A. When you introduce fractions to your class which of the following aspects do you use?

Please tick or write YES for any that apply:

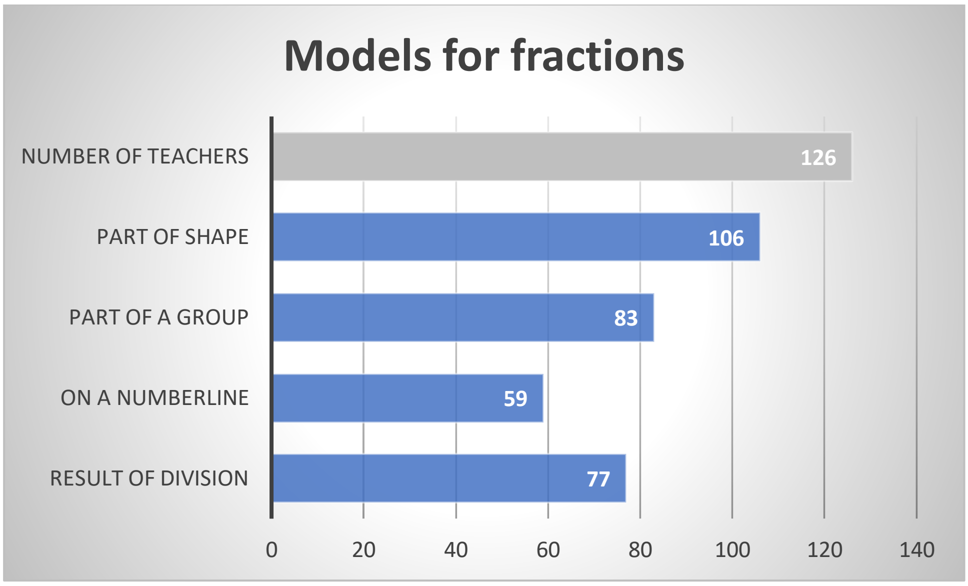
Part of a shape ***e.g. shade in a quarter of this circle***
Part of a group ***e.g. Find a quarter of 20p***
Fractions on a number line ***e.g. Where is two and a quarter on this number line?***
Fractions resulting from a division ***e.g. Share 6 apples between 4 people*** **or** ***Share 2 pies between 3 people***

B. When you introduce decimals to your class do you use any of the following contexts?

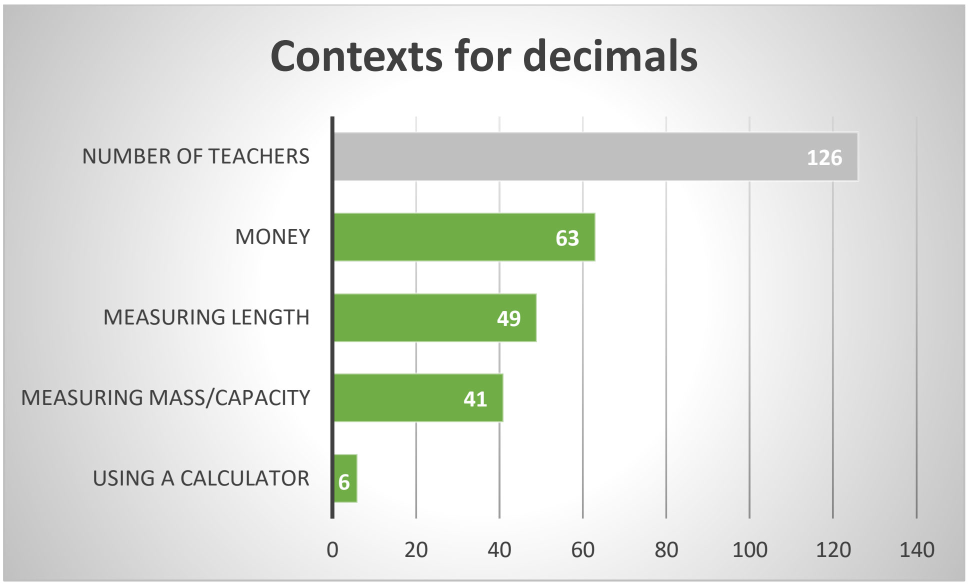
Please tick or write YES for any that apply:

Money
Measuring length
Measuring weight or capacity
Calculators

Across the whole age range, for fractions, part of a shape and part of a group were the main meanings or models used. Five teachers who were engaged in the NCETM Maths Hubs, added bar models to their list. Most teachers of younger children (aged 3 to 6) concentrated on sharing small groups of items with no 'remainder'; few of them used fractions on the number line.



We realised that we would need to unpack the meanings of fractions in considerable depth for the benefit of teachers as well as children, as we developed our tasks and examples.



We found that little use was made of measuring, even when introducing decimals. Calculators were not used by many teachers and their power as an investigative tool seemed not to be recognised.

We asked about the manipulatives and practical equipment that teachers used, with a suggested list of 20 items and space to add further examples. Pre-pandemic, we had planned to provide the questionnaires for teachers to complete in five minutes at the beginning of a staff meeting, so that we had whole-school responses. We would then have followed up with focus groups in a sample of schools, to find out more about how children had used the materials. Neither of these options was feasible while schools were coping with Covid. However, we were able to follow up a very small sample of questionnaires (7 teachers) with phone calls, to test out places where we realised the questionnaire may have been answered differently from our expectations.

We are therefore wary of reading too much into the responses we collected, but the results did confirm that many teachers used sets of cubes, counters and everyday objects, presumably mainly linked to 'sharing' activities. Base ten equipment was used by over half the teachers, but we realised this was probably more often to answer questions like 'What is half of 48?' rather than using the hundreds, tens and ones to represent ones, tenths and hundredths.

C. What practical apparatus/resources have the children in your class used to support their learning of fractions and decimals? Please tick or write YES for any that apply:

Cubes or counters	105	Fraction strips or walls	72
Bead strings	38	Fraction circles	53
Number lines	77	Pattern blocks	15
Cuisenaire rods	28	Calculators	8
Dienes' base 10 equipment (100s, 10s and 1s)	65	Measuring equipment e.g. tape measures, weighing scales	44
Place value charts (decimal)	47	10 frames (or Numicon)	46
Food or pretend food	77	Paper strips	72
Small toys	54	Play dough/Blutack/clay	42
Everyday objects	77	Area	4
Coins and notes	57	Arrays	23

Other items that were used by more than half the teachers were number lines, fraction strips and paper strips, and food. Although 49 teachers had said they used measuring as a context (see question B above), only 44 used measuring equipment.

Less use was made of practical resources as the children got older.

THE DEVELOPMENT OF EXAMPLES FOR TEACHERS

Our previous work with Oxford University Press on *Making Numbers* had given us the opportunity to discuss with teachers what worked in supporting their classroom practice. Many had said that they needed information about what children needed to learn and direct examples of possible activities that could achieve this. They also appreciated ideas of what children might actually do – including oral, practical and drawn or written examples to think about.

Our initial decisions when thinking about fractions and decimals were about the best learning trajectories to follow. These decisions took into account our reading, advice from our advisory group, and our work with children and teachers. We used various story contexts to sustain children's engagement and develop *Making Fractions*.

Making Fractions	
Chapter One: **Introduction**	*Importance of context and problems to solve. Why practical work? Building on children's informal knowledge of fractions; consolidating their skills and understanding of whole numbers: counting, comparison and composition.* *Usefulness of numbers 'in-between' whole numbers; linking fractions and decimals.*
Chapter Two **Halves and quarters** Building on children's early experience	Halves and then quarters. Importance of the nature of the whole. Which is more – one half or a quarter? Counting with fractions; using and writing the number symbols. Fractions of shapes. Fractions as a result of dividing: fractions of groups; exploring even *and* odd numbers and dividing by two. Not everything can be halved. Fractions in order. Contexts for mixed numbers. Making amounts up to six or more, with halves and quarters. Addition, subtraction; equivalence. Animation: Jam Tarts and Halves
Chapter Three **Thirds to thirteenths** Using families of fractions	Reviewing halves and quarters – include eighths – families of fractions; Introducing thirds and sixths as another family and compare with halves Number line as a model and locating fractions. Generalizing to show how we name fractions – thirteenths. Exploring the inverse relationship and equivalences. Looking at tenths; cutting fifths in half to make tenths. Counting, comparison and composition with these fractions. Building familiarity and fluency.

Chapter Four **Tenths and decimals** Starting from measurement	Measuring length. Starting with metres and moving to halves and then tenths. To be more accurate, we use smaller units of measurement. How does the calculator write ½? *Why* does the calculator write a half as 0.5? Look at sharing between 2. Saying and writing decimal numbers. Modelling with Dienes and beadstrings. Placing the numbers on a number line. Centimetres and tenths of a centimetre. Link between common fraction and decimal fraction notation. Tenths of a kilogram. Using a calculator to explore place value with tenths. Addition, subtraction; equivalence (beadstrings) Animation: Decimal Dragon
Chapter Five **Tenths and hundredths** Exploring decimals and measurement	Becoming even more accurate: hundredths as well as tenths. Equivalences. Saying and writing these. Rounding to the nearest tenth and to nearest whole number. Using metres and tenths of a metre, extending to metres, tenths and hundredths of a metre. Finding a tenth of 100, 1000; Dividing by ten. Finding how many tenths there are in numbers bigger than one. Money and decimals Common fraction to decimal equivalences e.g. $\frac{1}{4}$= 0.25, $\frac{2}{5}$ = 0.4
Chapter Six: **Looking Forward**	*Reminder of how each of these topics is linked to the others.* *Importance of having a problem to solve.*

The importance of story

Swan (2003) outlined the principles of design-based research, which focuses on changing classroom practice through the devising of tasks for children to learn from. He described a process of 'teaching for meaning', that can begin by presenting a problem for children to think about, to share ideas and perhaps to become aware that new learning is needed. Our intention was to introduce problems with a story wherever it seemed helpful – a context that children could imagine or could try out for themselves.

We developed two stop frame animations linked to our project, one which deals with sharing (and halves) and the other with measuring (and decimal numbers). The first story, linked to Chapter Two, shows four animal friends sharing different numbers of jam tarts, sometimes having to cut jam tarts in half to make the shares fair. They also show a fifth friend how to check how many jam tarts you have altogether. The story introduces mixed numbers in a straightforward way which our work in schools showed that children found very accessible.

The second story, linked to Chapter Five, stars Dragon, Sheep and Bee. Sheep helps measure the height of a bean and of Dragon in centimetres, and examines the weight of tins and packs of food in kilograms, using measurements to the nearest tenth. Both *Jam Tarts and Halves* and *Decimal Dragon* will be available free on the OxfordOwl website from January 2023.

We want children to be able to approach a problem with imagination and an open mind, willing to explore a situation: just to 'have a go'. We also want them to gradually build their skills and understanding, and to see the benefits of sometimes using systematic methods which allow them to see whether a pattern emerges.

The 'story' behind an activity could be something interesting that other children were trying to find out. For example, we found that activities such as measuring their own hops and jumps, gave children more motivation to measure accurately than the more traditional activity of measuring classroom furniture. This led them to be interested in using smaller fractions, becoming familiar with fractions as numbers on a scale, as well as checking with each other and discussing their results, working in pairs or threes.

How did we devise and trial our activities?

Owing to the pandemic, we had to find new ways of developing tasks and gathering information. Our initial plan had been to trial activities at a very early stage of development, using a group of teacher-volunteers to try them with their classes and report back for redrafting and retrialling. Trialling in schools was not often possible: schools were under a great deal of stress; many teachers were juggling having to provide for some children in school and some online, as well as coping with personal illness and family responsibilities.

We adopted a process of spending longer trying out any initial idea between ourselves; one person would write it up, then the other two would try it out and critique it, often drawing upon previous experience in the classroom and previous records of our own research. We also recruited small groups of children who were at home (sometimes with teacher-parents), to try anything that we had not taught before, until an activity was ready to write as pages for the book with draft illustrations and diagrams. Activities which we had not taught before, were sent out to volunteer teachers or to individual members of our advisory group for comment. When we were able to access schools again, we took activities to try with children and to collect examples of both their written work and the discussions they engaged in. Many of these examples are included in *Making Fractions.*

Working with children who were at home because their schools were closed was very interesting, as they were often in family groups. When we worked on *Making Numbers*, we were clear about the maths that each child would have been likely to have completed in the previous year, and we usually tried each activity with just one or two year groups. Working with children across the last eighteen months of lockdowns has meant some ideas have been tried with children over several year groups. We have sometimes been surprised at how enthusiastic older children have been with ideas that we thought would only appeal to younger ones. We have also been interested to see how competent young children have sometimes been with activities aimed at older children, especially in a story context. Our work with children has revealed that some have had a difficult time sustaining learning over the pandemic.

Making Fractions will help teachers in both assessing what children understand and helping to build their competence and confidence. Teachers can use the book to find engaging activities from different chapters for revision and consolidation. Our book is intended to support teachers in securing children's foundational knowledge of fractions and decimals. It encourages reasoning, problem solving and sense-making. Contexts such as sharing and measuring can inspire children to explore fractions and decimals in creative ways, deepening their understanding of these important numbers.

This main report

A GUIDE TO THE USE OF PRACTICAL TASKS AND MANIPULATIVES IN THE TEACHING OF FRACTIONS AND DECIMALS WITH CHILDREN AGED 3 TO 11

has two supplements:

REVIEW OF RESEARCH

EXAMPLES FOR TEACHERS

available on the Nuffield Foundation website

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